Consultation is fine, but no electronics, please.

The point values are to give you an idea of the relative importance or difficulty of the questions, but they are not the same as point values on quizzes or exams.

(1) (20 points) In $\mathbb{R}^2$, one basis is $\mathcal{B} = \{\begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}\}$ and another basis is $\mathcal{C} = \{\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\}$. Find the two basis change matrices $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$.

Solution. I will find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ first.

Method 1. Since $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \big[[\mathbf{b}_1]_\mathcal{C} \quad [\mathbf{b}_2]_\mathcal{C}\big] = \left[\begin{bmatrix}3\\1\end{bmatrix}_\mathcal{C} \quad \begin{bmatrix}1\\3\end{bmatrix}_\mathcal{C}\right]$, I need to solve the two linear systems

$$\begin{bmatrix}3\\1\end{bmatrix} = x_1\begin{bmatrix}1\\-1\end{bmatrix} + x_2\begin{bmatrix}1\\1\end{bmatrix} \quad\text{and}\quad \begin{bmatrix}1\\3\end{bmatrix} = y_1\begin{bmatrix}1\\-1\end{bmatrix} + y_2\begin{bmatrix}1\\1\end{bmatrix}.$$

If you spot the solutions $\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix}$ immediately, use them. If not, set up the augmented matrices (combined, for convenience) and reduce:

$$\begin{bmatrix}1 & 1 & | & 3 & 1\\ -1 & 1 & | & 1 & 3\end{bmatrix} \to \begin{bmatrix}1 & 0 & | & 1 & -1\\ 0 & 1 & | & 2 & 2\end{bmatrix}.$$

Method 2. Set up $[\mathbf{c}_1 \quad \mathbf{c}_2 \quad |\mathbf{b}_1 \quad \mathbf{b}_2] = \begin{bmatrix}1 & 1 & | & 3 & 1\\ -1 & 1 & | & 1 & 3\end{bmatrix} \to \begin{bmatrix}1 & 0 & | & 1 & -1\\ 0 & 1 & | & 2 & 2\end{bmatrix}$.

Either way, the result is that $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix}1 & -1\\ 2 & 2\end{bmatrix}$.

(2) (10 points) Let $\mathbf{x} \in \mathbb{R}^2$. Use one of your basis change matrices to convert the coordinate vector $[\mathbf{x}]_\mathcal{B} = \begin{bmatrix}4\\-1\end{bmatrix}$ to the coordinate vector $[\mathbf{x}]_\mathcal{C}$, *without finding the vector* $\mathbf{x}$.

Solution. $[\mathbf{x}]_\mathcal{C} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_\mathcal{B} = \begin{bmatrix}1 & -1\\ 2 & 2\end{bmatrix}\begin{bmatrix}4\\-1\end{bmatrix} = \begin{bmatrix}5\\6\end{bmatrix}$.

TURN OVER FOR MORE! MORE!

(3) (20 points) In $\mathbb{P}_2$, one basis is $\mathcal{E} = \{1, x, x^2\}$ and another is $\mathcal{A} = \{x+x^2, 1+x^2, 1+x\}$. Find the two basis change matrices $\underset{\mathcal{E}\leftarrow\mathcal{A}}{P}$ and $\underset{\mathcal{A}\leftarrow\mathcal{E}}{P}$. Hint: One of them is easy.

Solution. First, which one is easy? Look at the formulas:

$$\underset{\mathcal{E}\leftarrow\mathcal{A}}{P} = \begin{bmatrix}[\mathbf{a}_1]_{\mathcal{E}} & [\mathbf{a}_2]_{\mathcal{E}} & [\mathbf{a}_3]_{\mathcal{E}}\end{bmatrix}, \qquad \underset{\mathcal{A}\leftarrow\mathcal{E}}{P} = \begin{bmatrix}[\mathbf{e}_1]_{\mathcal{A}} & [\mathbf{e}_2]_{\mathcal{A}} & [\mathbf{a}_3]_{\mathcal{A}}\end{bmatrix}.$$

We can read off $\mathcal{E}$-coordinates, $[\ ]_{\mathcal{E}}$, easily, so I choose the easy case:

$$\underset{\mathcal{E}\leftarrow\mathcal{A}}{P} = \begin{bmatrix}[x+x^2]_{\mathcal{E}} & [1+x^2]_{\mathcal{E}} & [1+x]_{\mathcal{E}}\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}0\\1\\1\end{bmatrix} & \begin{bmatrix}1\\0\\1\end{bmatrix} & \begin{bmatrix}1\\1\\0\end{bmatrix}\end{bmatrix} = \begin{bmatrix}0 & 1 & 1\\1 & 0 & 1\\1 & 1 & 0\end{bmatrix}.$$

Then you can invert this to get $\underset{\mathcal{A}\leftarrow\mathcal{E}}{P}$, or you can compute the $\mathcal{A}$-coordinate vectors of the $\mathcal{E}$-basis elements. Either one is good. I choose matrix inversion:

$$\begin{bmatrix}0 & 1 & 1 & | & 1 & 0 & 0\\1 & 0 & 1 & | & 0 & 1 & 0\\1 & 1 & 0 & | & 0 & 0 & 1\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0 & | & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\0 & 1 & 0 & | & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\\0 & 0 & 1 & | & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2}\end{bmatrix},$$

so I conclude that

$$\underset{\mathcal{A}\leftarrow\mathcal{E}}{P} = \begin{bmatrix}-\frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\\\frac{1}{2} & \frac{1}{2} & -\frac{1}{2}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-1 & 1 & 1\\1 & -1 & 1\\1 & 1 & -1\end{bmatrix}.$$

If you chose the easier matrix (or the harder matrix) and didn't do the other one, you did well enough. The purpose of this question was (a) to work with a vector space that isn't $\mathbb{R}^n$, and (b) to see which matrix is easier.