Class Problem 3 Math 304-06 Nov. 3, 2023 Solutions

Matrix $A = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & 3 \end{bmatrix}$.

(1) Find the eigenvalues (with multiplicities) and corresponding eigenvectors of A.

We calculate $|A - \lambda I| = \begin{vmatrix} 5-\lambda & 0 & 0 \\ 1 & 5-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = (5-\lambda)^2(2-\lambda) = 0$, whose zeros ("roots") are $\lambda = 5, 5, 2$. This gives the eigenvalues. (If you just read them off the diagonal, that's also correct.)

Eigenvalue 5: The eigenspace is $\mathrm{Nul}(A - 5I)$, which we find by row reduction.

$$A - 5I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the eigenvectors are the vectors that satisfy $x_1 = 0$ and $x_2 - x_3 = 0$. The eigenspace is $E_5 = \{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 = 0,\ x_2 = x_3\} = \{ \begin{bmatrix} 0 \\ x_3 \\ x_3 \end{bmatrix} : x_3 \in \mathbb{R}\} = \mathrm{span}\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \}$. The eigenvectors are all the nonzero vectors in E_5.

Eigenvalue 2: The eigenspace is $\mathrm{Nul}(A - 2I)$, which we find by row reduction.

$$A - 5I = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the eigenvectors are the vectors that satisfy $x_1 = 0$ and $x_2 = 0$. The eigenspace is $E_2 = \{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 = 0,\ x_2 = 0\} = \{ \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} : x_3 \in \mathbb{R}\} = \mathrm{span}\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$. The eigenvectors are all the nonzero vectors in E_2.

(2) Find a basis for $\mathbb{R}^3$ that consists of eigenvectors of A. How do you know it's a basis?

There is no such basis, because the sum of dimensions of the two eigenspaces is 2, which is less than $\dim \mathbb{R}^3 = 3$.

(3) Prove that B is invertible. (Hint: Determinant!)

$\det B = 1 \cdot 2 \cdot 3 + 0 + 0 - 0 - 0 - 0 \neq 0$, so B is invertible by the determinant test.

TURN OVER FOR MORE! MORE!

(4) Are A and $B^{-1}AB$ similar? Circle one: <u>Yes</u> No

Reason (not required by the question, but of course you should know): B is invertible and any matrix $B^{-1}AB$ with invertible B is similar to A by definition of similarity. (The book's P is my B^{-1}; this is just a change of notation.)

(5) Find the eigenvalues (with multiplicities) and corresponding eigenvectors of $B^{-1}AB$.

The eigenvalues are the same as those of A because of similarity, i.e., $5, 5, 2$.

The eigenvectors require some work. One method is to compute the matrix $B^{-1}AB$ and find its eigenvectors in the usual way. I will use a different method to illustrate properties of similarity.

If $\mathbf{x}$ is an eigenvector of $B^{-1}AB$ with eigenvalue 5, then $(B^{-1}AB)\mathbf{x} = 5\mathbf{x}$. Multiply on the left by B to get

$$B(B^{-1}AB\mathbf{x}) = B(5\mathbf{x}),$$

so $(BB^{-1})A(B\mathbf{x}) = 5(B\mathbf{x})$ by matrix algebra. This simplifies to

$$A(B\mathbf{x}) = 5(B\mathbf{x}).$$

That is, $B\mathbf{x}$ is an eigenvector of A with eigenvalue 5. Thus, $B\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Then we solve this linear system for $\mathbf{x}$.

If $\mathbf{y}$ is an eigenvector of C with eigenvalue 2, then $C\mathbf{y} = 2\mathbf{y}$. By similar calculations, $B\mathbf{y}$ is an eigenvector of A with eigenvalue 2. Thus, $B\mathbf{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. We solve for $\mathbf{y}$.

We can solve both at the same time by setting up a doubly augmented matrix (as we've done before). The matrix is

$$\left[\begin{array}{ccc|cc} 1 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 0 & 3 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|cc} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 \\ 1 & 0 & 1 & 1/3 & 1/3 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|cc} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 \\ 0 & -2 & 1 & 1/3 & 1/3 \end{array}\right] \to \left[\begin{array}{ccc|cc} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 7/6 & 1/3 \end{array}\right].$$

Thus, $\mathbf{x} = \begin{bmatrix} -1 \\ 1/2 \\ 7/6 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 0 \\ 0 \\ 1/3 \end{bmatrix}$ are the desired eigenvectors (or you can clear fractions, or multiply by any nonzero scalar).

Note that I only gave one eigenvector (a basis vector), since the question says "Find eigenvectors" and both eigenspaces have dimension 1. In general "find eigenvectors" means find a spanning set: a basis, or the whole eigenspace if you prefer. (The exams probably won't use exactly this language.)

The principle here is that the eigenvectors of $B^{-1}AB$ are related to those of A in a particular way that I can use to find them from the eigenvectors of A.