(1) The linear transformation $T : \mathbb{P}_1 \to \mathbb{P}_1$ is defined by $T(p(x)) = \dfrac{d}{dx}p(x) + p(0)x$. Find the eigenvalues and eigenspaces of T. (Do not convert T into a matrix.)

Solution:

Eigenvalues: We want a scalar λ such that $T(p(x)) = \lambda p(x)$ for some polynomial $p(x) \in \mathbb{P}_1$. *That is what an eigenvalue means.*

Using the formula for T, that means $p'(x) + p(0)x = \lambda p(x)$.

To use this we need to write the (unknown) polynomial $p(x) = a_0 + a_1 x$. *That is a standard technique.* Then $p'(x) = a_1$ and $p(0) = a_0$, so we want

$$a_1 + a_0 x = \lambda(a_0 + a_1 x).$$

Two polynomials are equal (as polynomials) if and only if they have the same coefficients. Therefore, we must have

$$\lambda a_0 - a_1 = 0 \text{ and } \lambda a_1 - a_0 = 0.$$

This is a homogeneous linear system with variables a_0, a_1, whose coefficient matrix is

$$\bar{A} = \begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix}.$$

We want to find λ such that $\bar{A}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has nontrivial solutions. (That is, the null space of A is not the zero subspace.) This happens if and only if $\bar{A}$ is singular. Our technique for deciding this is to set $\det \bar{A} = 0$ and solve for λ, the same as when we find eigenvalues of a matrix.

This means we want to solve

$$\begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - 1 = 0$$

with solution $\lambda = \pm 1$. Those are the eigenvalues of T.

Eigenvectors: For eigenvalue $\lambda = 1$ we want to find all $p(x)$ such that $T(p(x)) = 1p(x)$; that is, $a_1 + a_0 x = 1(a_0 + a_1 x)$. *The coefficients of powers of x on both sides must be equal*, so $a_1 = a_0$ and $a_0 = a_1$. This is a homogeneous linear system with two equations and coefficient matrix $\bar{A} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ (putting $\lambda = 1$). Then $\operatorname{Nul} \bar{A} = \operatorname{span}\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$. The eigenspace is a subspace of $\mathbb{P}_1$ so it consists of polynomials $a_0 + a_1 x$ where $\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \in \operatorname{span}\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$. That is, $\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_0 \end{bmatrix}$. Thus, $p(x) = a_0 + a_0 x = a_0(1 + x)$ and the eigenspace is $\operatorname{span}\{1 + x\}$. A basis is $\{1 + x\}$.

For eigenvalue $\lambda = -1$ we want to find all $p(x)$ such that $T(p(x)) = -p(x)$. The calculation is similar. The matrix $\bar{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Its null space is $\operatorname{Nul} \bar{A} = \operatorname{span}\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$.

The eigenspace consists of polynomials $a_0 + a_1 x$ where $\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \in \text{span}\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$. That is, $\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} -a_1 \\ a_1 \end{bmatrix}$. Thus, $p(x) = a_1(-1 + x)$ and the eigenspace is $\text{span}\{-1 + x\}$. A basis is $\{-1 + x\}$.

(2) A linear transformation $T : \mathbb{P}_1 \to \mathbb{P}_1$ has matrix $[T]_{\mathcal{C}} = \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}$ with respect to the basis $\mathcal{C} = \{1 - x, 3 + x\}$. Find the value of $T(2x)$.

Solution:

This involves three steps. (Taking shortcuts can easily lead to wrong answers.)

First step: Find $[2x]_{\mathcal{C}}$. This is $\begin{bmatrix} a \\ b \end{bmatrix}$ such that $a(1 - x) + b(3 + x) = 2x$. Thus, $a + 3b = 0$ and $-a + b = 2$. Solution: $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -3/2 \\ 1/2 \end{bmatrix}$.

Second step: Compute $[T(2x)]_{\mathcal{C}} = [T]_{\mathcal{C}} \, [2x]_{\mathcal{C}} = \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -3/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ -5/2 \end{bmatrix}$.

Final step: Find $T(2x)$ from its coordinate vector $[T(2x)]_{\mathcal{C}}$. Thus,

$$T(2x) = 1(1 - x) - \frac{5}{2}(3 + x) = -\frac{13}{2} - \frac{7}{2}x.$$