(1) (20 fictional points) Diagonalize the matrix $A = \begin{bmatrix} 6 & -5 \\ 5 & 0 \end{bmatrix}$.

First, find the eigenvalues.

$$\det(A - \lambda I) = \begin{vmatrix} 6-\lambda & -5 \\ 5 & 0-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 25 = 0$$

gives $\lambda = \dfrac{6 \pm \sqrt{36 - 4 \cdot 25}}{2} = 3 \pm 4i$. Note that you can now write the diagonal matrix: $D = \begin{bmatrix} 3+4i & 0 \\ 0 & 3-4i \end{bmatrix}$.

Next, find the eigenspace basis for $\lambda = 3 + 4i$. We want the null space of

$$A - (3+4i)I = \begin{bmatrix} 3-4i & -5 \\ 5 & -3-4i \end{bmatrix}.$$

Carry out the usual row operations to get reduced row echelon form:

$$\begin{bmatrix} 3-4i & -5 \\ 5 & -3-4i \end{bmatrix} \to \begin{bmatrix} 3-4i & -5 \\ 1 & -\frac{3+4i}{5} \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{3+4i}{5} \\ 3-4i & -5 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & -\frac{3+4i}{5} \\ 0 & -5 + (3-4i)\frac{3+4i}{5} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{3+4i}{5} \\ 0 & 0 \end{bmatrix}$$

Therefore $\mathbf{x}$ is in the eigenspace if and only if $x_1 = \frac{3+4i}{5} x_2$, i.e., $\mathbf{x} = \begin{bmatrix} \frac{3+4i}{5} \\ 1 \end{bmatrix} x_2$. The basis vector we want is $\begin{bmatrix} \frac{3+4i}{5} \\ 1 \end{bmatrix}$, or any nonzero scalar multiple, such as $\begin{bmatrix} 3+4i \\ 5 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ \frac{3-4i}{5} \end{bmatrix}$.

The calculation for $\lambda = 3 - 4i$ is similar. The eigenvector is $\begin{bmatrix} \frac{3-4i}{5} \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 3-4i \\ 5 \end{bmatrix}$, etc.

Finally, you write the matrix $P = \begin{bmatrix} \frac{3+4i}{5} & 3-4i \\ 1 & 5 \end{bmatrix}$. Two things to notice: It doesn't matter which scalar multiple you choose for each column. Also, the first column's eigenvector must correspond to the first eigenvalue, the second to the second.

The solution to "Diagonalize" is to find D and P. No further calculation is necessary.

(2) (2 fictional points) What do you notice about the two eigenvalues? Also, what about the two eigenvectors?

The main thing to notice is that the complex eigenvalues are complex conjugates and *also* their eigenvectors are complex conjugates. This is a general phenomenon for a real matrix with complex eigenvalues and you can use it to simplify your work. If you find eigenvectors for one complex eigenvalue, you can merely conjugate them to get the eigenvectors for the conjugate eigenvalue.