QUIZ 11 Math 304-06 Solution Print your name RESOLVER
Nov. 21, 2023

(1) (10 points) In $\mathbb{R}^2$ the set $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2\}$, where $\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$, is an orthonormal basis. Find the expression of $\mathbf{x} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ as a linear combination of this basis, *using the orthonormality of* $\mathcal{U}$.

Solution. Since the basis is orthonormal we can find the coefficients as follows:

$$\mathbf{x} = (\mathbf{x} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{x} \cdot \mathbf{u}_2)\mathbf{u}_2 = \frac{7}{\sqrt{2}}\mathbf{u}_1 + \frac{-3}{\sqrt{2}}\mathbf{u}_2.$$

If you forgot how to use normality you get 8 points for the following calculation:

$$\mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{x} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 = \frac{7/\sqrt{2}}{1}\mathbf{u}_1 + \frac{-3/\sqrt{2}}{1}\mathbf{u}_2 = \frac{7}{\sqrt{2}}\mathbf{u}_1 + \frac{-3}{\sqrt{2}}\mathbf{u}_2.$$

If you used a different method such as solving two equations to get the coefficient you get 5 points, but you didn't really answer the question.

(2) (15 points) Diagonalize the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$.

Solution. First find the eigenvalues:

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 0-\lambda \end{vmatrix} = (1-\lambda)(-\lambda) - 4 = \lambda^2 - \lambda - 4 = 0,$$

giving eigenvalues $\lambda = \frac{1+\sqrt{17}}{2}, \frac{1-\sqrt{17}}{2}$. I assumed you can do this; the credit is entirely for getting the P matrix (9 points) and D matrix (6 points).

For P we need the eigenvectors. For $\lambda = \frac{1+\sqrt{17}}{2}$ we want

$$\text{Nul}\begin{bmatrix} 1 - \frac{1+\sqrt{17}}{2} & 2 \\ 2 & -\frac{1+\sqrt{17}}{2} \end{bmatrix} = \text{Nul}\begin{bmatrix} \frac{1-\sqrt{17}}{2} & 2 \\ 2 & -\frac{1+\sqrt{17}}{2} \end{bmatrix} = \text{Nul}\begin{bmatrix} 1 & -\frac{1+\sqrt{17}}{4} \\ 0 & 0 \end{bmatrix}.$$

Therefore, $x_1 = \frac{1+\sqrt{17}}{4}x_2$ so the eigenvector is $\begin{bmatrix} \frac{1+\sqrt{17}}{4} \\ 1 \end{bmatrix}$.

For $\lambda = \frac{1-\sqrt{17}}{2}$ the calculation is similar and the eigenvector is $\begin{bmatrix} \frac{1-\sqrt{17}}{4} \\ 1 \end{bmatrix}$.

This is all the information we need for diagonalization. The answer is

$$D = \begin{bmatrix} \frac{1+\sqrt{17}}{4} & 0 \\ 0 & \frac{1-\sqrt{17}}{4} \end{bmatrix}, \quad P = \begin{bmatrix} \frac{1+\sqrt{17}}{4} & 1 \\ 1 & \frac{1-\sqrt{17}}{4} \end{bmatrix}.$$

You do not need to invert P unless it is specifically asked for. If your calculations are correct (I checked mine!), then $A = PDP^{-1}$ (equivalently: $D = P^{-1}AP$, and $AP = PD$) is automatically true.