QUIZ 2 Math 304-06 Print your name ________________________________
Sept. 8(11), 2023

Show full work for each question. You must show work for credit.
No consultation!—that includes no electronics.

(1) This matrix $A = \begin{bmatrix} 0 & 2 & 2 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$ is the coefficient matrix of a *homogeneous* linear system. Solve the system by finding the reduced echelon form of A. For full credit, express your answer in vector form with the variables x_1, x_2, x_3, x_4 separated.

6 points for reduced row echelon form (RREF):

$$a \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

6 points for expressing the solution in terms of free variables. The equations are

$$x_1 + x_3 = 0$$
$$x_2 + x_3 + 2x_4 = 0.$$

From the RREF we see that the first two columns have pivot positions so their variable are basic, and the other two variables are free variables. Therefore the solution is

$$x_1 = -x_3$$
$$x_2 = -x_3 - 2x_4$$

x_3, x_4 are free variables. (You must say this.)

3 points for expressing the solution in vector form:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 - 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}.$$

(2) Decide whether these four vectors are linearly dependent or independent.

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{a}_4 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}.$$

If they are linearly dependent, show a specific linear dependence relation (with the coefficients).

5 points for the correct answer that they are linearly dependent, but with a deduction for an invalid reason.

5 points for showing a linear dependence relation among the four vectors, which can be either of the following (and there are other possibilities):

$$0\mathbf{a}_1 + 0\mathbf{a}_2 + 1\mathbf{a}_3 + 0\mathbf{a}_4 = \mathbf{0},$$
$$1\mathbf{a}_1 - 1\mathbf{a}_2 + 0\mathbf{a}_3 + 1\mathbf{a}_4 = \mathbf{0}.$$

In a question like this, we are not asking about individual vectors. We are asking about all the vectors together, as a set of vectors. An individual vector can't be linearly dependent or independent, it has no meaning. A set of one vector can be, but here we have four vectors. Also note that the linear dependence relation is an *equation*.

(3) A transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 - x_1 \\ x_1 \end{bmatrix}.$$

(a) Evaluate $T\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)$.

<u>5 points</u> for $T\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3+1 \\ 1-3 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$.

(b) Prove that T is a linear transformation using the definition of linear transformations.

<u>10 points</u> for a correct proof, <u>3 points</u> for no proof but showing you know the two properties in the definition.

We have to prove two properties: $T(c\mathbf{x}) = cT(\mathbf{x})$ and $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ for *any* vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ in the domain $\mathbb{R}^2$ and *any* scalar $c \in \mathbb{R}$. If you only use specific vectors, you do not have a proof.

For the first property: $T(c\mathbf{x}) = T\left(\begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}\right) = \begin{bmatrix} cx_1 + cx_2 \\ cx_2 - cx_1 \\ cx_1 \end{bmatrix} = c\begin{bmatrix} x_1 + x_2 \\ x_2 - x_1 \\ x_1 \end{bmatrix} = cT(\mathbf{x})$.

For the second property:

$$\begin{aligned} T(\mathbf{x}+\mathbf{y}) = T\left(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}\right) &= \begin{bmatrix} (x_1+y_1) + (x_2+y_2) \\ (x_2+y_2) - (x_1+y_1) \\ (x_1+y_1) \end{bmatrix} \\ &= \begin{bmatrix} x_1 + x_2 \\ x_2 - x_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} y_1 + y_2 \\ y_2 - y_1 \\ y_1 \end{bmatrix} = T(\mathbf{x}) + T(\mathbf{y}). \end{aligned}$$

That completes the proof.

A note on 3(b): You should know these two properties as we will need them later. You will possibly see them on the test. You may be asked to prove a certain transformation is linear. I put this on the quiz to make sure you think about it.