QUIZ 7 Math 304-06 Print your name <u>Warlock Witch</u>
Oct. 23, 2023

(1) (3 points each) A, B are $n \times n$ matrices. Circle your answer: true or false.

(a) True <u>False</u> $\det A = \det(A^{-1})$.

(b) <u>True</u> False $\det A = \det(A^T)$.

(c) <u>True</u> False $\det A = 1$ if $A = I_n$.

(d) True <u>False</u> $\det A = 1$ is impossible if $A \neq I_n$.

(e) <u>True</u> False $\det(AB) = (\det A)(\det B)$.

(f) <u>True</u> False If E is the elementary matrix that interchanges rows i and j (where $i \neq j$), then $\det E = -1$.

(2) (12 points) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ has the matrix $[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$ with respect to the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ where $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Problem: Evaluate $T(\mathbf{x})$ where $\mathbf{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

Solution. The general formula we use is about coordinate vectors:

$$[T(\mathbf{x})]_{\mathcal{B}} = [T]_{\mathcal{B}}\, [\mathbf{x}]_{\mathcal{B}}.$$

We are given $[T(\mathbf{x})]_{\mathcal{B}}$, so we should find $[\mathbf{x}]_{\mathcal{B}}$. It's easy to see (if someone tells you; or see below) that $\mathbf{x} = 1 \begin{bmatrix} 3 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; that means the coordinate vector of $\mathbf{x}$ is $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Applying the general formula,

$$[T(\mathbf{x})]_{\mathcal{B}} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

Now we reverse the coordinate process, extracting the vector $T(\mathbf{x})$ whose coordinate vector is $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$. That vector is $4 \begin{bmatrix} 3 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \end{bmatrix}$. In other words, $T(\mathbf{x}) = \begin{bmatrix} 12 \\ 0 \end{bmatrix}$.

How to find the coordinate vector $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$? We need a_1, a_2 such that $a_1 \begin{bmatrix} 3 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \mathbf{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. We can set up a linear system with augmented matrix

$$\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

giving the solution $a_1 = 1,\ a_2 = 1$.