(1) (20 points) Find the eigenvalues of A. For each eigenvalue, find a basis for the eigenspace.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

Solution: I give it in 3 parts.

Part 1 (8 points): The eigenvalues are 1, 1, 3, or you could say, 1 twice and 3, or other ways. You have to show the multiplicity for full credit (that is, for a multiple eigenvalue; not mentioning multiplicity means the multiplicity is 1).

Part 2 (6 points): Eigenbasis for 1**.** We want $\mathrm{Nul}(A - 1I) = \mathrm{Nul} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix}$.

The matrix reduces to $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. That means $x_3 = 0$, so the eigenspace is

$$E_1 = \{\mathbf{x} \in \mathbb{R}^3 : x_3 = 0\} = \{ \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} : x_1, x_2 \in \mathbb{R}\} = \mathrm{span}\{\mathbf{e}_1, \mathbf{e}_2\}.$$

This is not the final answer because the span is a subspace, not a basis, so the answer is that $\{\mathbf{e}_1, \mathbf{e}_2\}$ (note that I use set braces; the basis is a set) is a basis for the eigenspace.

Part 3 (6 points): Eigenbasis for 5**.** We want $\mathrm{Nul}(A-5I) = \mathrm{Nul} \begin{bmatrix} -4 & 0 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

The matrix reduces to $\begin{bmatrix} 1 & 0 & -1/4 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{bmatrix}$. That means $x_1 = x_2 = \frac{1}{4}x_3$, so the eigenspace is

$$E_5 = \{ \begin{bmatrix} \frac{1}{4}x_3 \\ \frac{1}{4}x_3 \\ x_3 \end{bmatrix} : x_3 \in \mathbb{R}\} = \mathrm{span}\{ \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ 1 \end{bmatrix} \}.$$

The answer is that a basis is $\{ \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ 1 \end{bmatrix} \}$, or if you prefer, $\{ \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \}$ or any other (nonzero) scalar multiple.

PLEASE TURN OVER FOR QUESTION 2

(2) (10 points) Are the matrices B and C similar? (5 points) Give a valid reason and circle your answer (5 points):

Yes   <u>No</u>

The matrices: $$B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}.$$

Solution: We prove that B and C don't have the same eigenvalues, so they can't be similar. B has eigenvalues 2 and 3.

Method 1: We find the eigenvalues of C. The characteristic polynomial is $p_C(\lambda) = \det \begin{bmatrix} 2-\lambda & 4 \\ 1 & 1-\lambda \end{bmatrix} = \lambda^2 - 3\lambda - 2$. Then $p_C(\lambda) = 0$ gives

$$\lambda = \frac{3 \pm \sqrt{17}}{2}.$$

They are not the eigenvalues of B.

Method 2: We can see that 2 is not an eigenvalue of C because $|C - 2I| = \begin{vmatrix} 0 & 4 \\ 1 & -1 \end{vmatrix} = -4 \neq 0$. Or, you can prove that 3 is not an eigenvalue of C. Either way is sufficient to prove B and C are not similar.