QUIZ 9 Solution   Math 304-06   Print your name Wizard of Algebra

Nov. 13, 2023

The vector space $\mathbb{P}_1$ has a basis $\mathcal{B} = \{x+1, x-1\}$. The linear transformation $T : \mathbb{P}_1 \to \mathbb{P}_1$ is defined by $T(p(x)) = \dfrac{d}{dx}p(x)$.

(1) (5 points) Evaluate $\dfrac{1}{3-4i}$ as a complex number, where $i = \sqrt{-1}$.

Solution. Since $1/z = \dfrac{\bar{z}}{|z|^2}$ for a complex number z, we can immediately write $\dfrac{1}{3-4i} = \dfrac{3+4i}{25}$.

Alternatively, write $\dfrac{1}{3-4i} = x+yi$ and solve for x and y using $1 = (3-4i)(x+yi) = (3x+4y) + (3y-4x)i$, giving $3y - 4x = 0$ and $3x + 4y = 1$, from which one can deduce that $x = \frac{3}{25}$ and $y = \frac{4}{25}$.

(2) (15 points) Find the matrix $[T]_{\mathcal{B}}$ of T with respect to basis $\mathcal{B}$.

Solution. The general formula says

$$[T]_{\mathcal{B}} = \left[[T(\mathbf{b}_1)]_{\mathcal{B}} \quad [T(\mathbf{b}_2)]_{\mathcal{B}}\right] = \left[[T(x+1)]_{\mathcal{B}} \quad [T(x-1)]_{\mathcal{B}}\right].$$

The next step is to calculate $T(x+1) = \frac{d}{dx}(x+1) = 1$ and $T(x-1) = \frac{d}{dx}(x+1) = 1$. The third step is to calculate $[1]_{\mathcal{B}} = \begin{bmatrix} a \\ b \end{bmatrix}$, which means to solve $1 = a(x+1) + b(x-1)$; that gives $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$. The fourth step is to put this into the matrix:

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & -1/2 \end{bmatrix}.$$