Supplement on Sets and Functions

1. Logic

A (logical) statement is something that can be true or false. For example, "All rocks are red" is a statement, but it happens to be false. "Some rocks are red" is a different statement and it happens to be true. "Many rocks are red" is a bit of a problem for logic, because I don't know exactly what "many" means; it could mean many (sic) things such as "hundreds of rocks are red" (true, I'm sure) or "40% of all rocks are red" (false on the Earth, possibly true on Mars), etc. "A cube" is not a statement since it's meaningless to say it is true or false.

Suppose P and Q are statements. We can combine them into compound statements, such as:

A. "P and Q" is the statement that both P and Q are true.
B. "P or Q" is the statement that P or Q *or both* are true. Note that "or" does not mean only one is true; it could be one or both. That is the meaning in logic. (In ordinary speech "or" is ambiguous: it could mean only one is true, or it could mean one or both are true. Logic is strict about precise meanings.)
C. "If P, then Q" or "P implies Q" means that if we know P is true, then we can be sure Q is true. If P is not true, we can't be sure about Q unless we have other information.
D. "Not P" means P is false. (We call "Not P" the *negation* of P.)

2. Sets

I assume everyone knows what a set is. Here are some reminders and notations. I use capital letters W, X, Y, Z for sets and lower-case letters w, x, y, z for elements of sets.

Set definition: The set of all objects that have a certain property is written

$$\{x : x \text{ has the property}\}.$$

We read this as follows: "The set of all x such that x has the property". You put in the property you want. For example, if the property is being a solution to $Ax = b$, we would write $\{x : x \text{ is a solution of } Ax = b\}$, but a simpler way is

$$\{x : Ax = b\}.$$

Usually, to be clear, we want to say that x must be in some big set, say U; then we write $\{x \in U : x \text{ has the property}\}$. For example,

$$\{x \in \mathbb{R}^n : Ax = b\}.$$

Intersection: $X \cap Y = \{x : x \in X \text{ and } x \in Y\}$.
Union : $X \cup Y = \{x : x \in X \text{ or } x \in Y\}$. (Remember what "or" means in logic.)

3. Functions

A function is a rule for taking an element of one set, say X, and producing an element of another set, say Y (but it could be the same set!). We call X the *domain* of the function, Y the *codomain*, and we give a name to the function so we can talk about it, say f (surprise?). Then we say f is a function from X to Y and we write $f : X \to Y$. In calculus you see functions like $y = x^2$, where the function is $f(x) = x^2$, the domain is $\mathbb{R}$, and the codomain is also $\mathbb{R}$.

Summary about functions:

Function: $f : X \to Y$ is any rule that, given an element $x \in X$, gives back a unique element $y \in Y$. We write $y = f(x)$. (Usually the rule is a formula.)

Domain: X is the domain of this function. It is where the inputs are.

Codomain: Y is the codomain of this function. It is where the values are.

The domain and codomain may be the same set or different sets.

Range or Image: $\mathrm{Range}\,(f)$ is the set of all the values that $f(x)$ can possibly take on, if you give it every element of the domain. Formally,

$$\begin{aligned}\mathrm{Range}\,(f) &= \{y : \text{there is an } x \in X \text{ such that } y = f(x)\} \\ &= \{f(x) : x \in X\}.\end{aligned}$$

The range may be the whole codomain, or it may be only a proper subset.

4. Functions in linear algebra

Here is a function we use often in linear algebra. Let A be an $m \times n$ matrix. Then $f(\mathbf{x}) = A\mathbf{x}$ is a function defined for all $\mathbf{x} \in \mathbb{R}^n$; we write $f : \mathbb{R}^n \to \mathbb{R}^m$. That is, the domain is $\mathbb{R}^n$; the function takes any element of $\mathbb{R}^n$ and gives us an element of $\mathbb{R}^m$ (the codomain).

Example. We do not expect to get every single element of $\mathbb{R}^m$ (the codomain) as a value of the function (i.e., as a value of $A\mathbf{x}$). For example, let O be the zero matrix (all 0's); then $A\mathbf{x} = \mathbf{0} \in \mathbb{R}^m$ for every $\mathbf{x}$, which means the function defined by $f(\mathbf{x}) = O\mathbf{x}$ misses almost all vectors in the codomain.

Returning to the general picture with $f(\mathbf{x}) = A\mathbf{x}$ where A is an arbitrary $m \times n$ matrix: There is a name for the set of vectors in the domain that we do get: it is the *range* or *image* of the function and its definition is

$$\mathrm{Range}\,(f) = \{\mathbf{y} \in \mathbb{R}^m : \text{there is a vector } \mathbf{x} \in \mathbb{R}^n \text{ such that } \mathbf{y} = A\mathbf{x}\}.$$

A simpler way to write this is

$$\mathrm{Range}\,(f) = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\}.$$

Example. For the function f given by $f(\mathbf{x}) = O\mathbf{x}$, the only value is the zero vector, $\mathbf{0}$. Then $\mathrm{Range}\,(f) = \{\mathbf{0}\}$, which is a small subset of the codomain, $\mathbb{R}^m$.