Math 304 2004/3/28 **TEST II** Name ________________________

- Show all your work for each problem; show enough work to fully justify your answer.
- All numerical answers should be in terms of actual numbers.
- Start each numbered problem on a *fresh page.*
- Hand in *both* this paper and test booklet.
- $\mathbf{P}_n$ is the set of all polynomials whose degree is no greater than n.

1. [Points: 20]
   (a) Show that the ordered set
   $$B = \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$$
   is a basis of $\mathbb{R}^3$.

   (b) Find the coordinate vector of $\mathbf{z} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ with respect to B.

   (c) Find the coordinate vector of $\mathbf{z}$ with respect to the ordered basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$.

2. [Points: 20] Consider the set $S = \{\mathbf{v}_1, \mathbf{v}_2\}$, where
   $$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}.$$

   (a) Is $\mathbf{w} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$ in Span(S)?

   (b) Which vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ are in Span(S)? For your answer, find the equation(s) that all such vectors must satisfy.

   (c) Find a basis for Span(S).

3. [Points: 10] Suppose A is an invertible $n \times n$ matrix. Prove that, if $f_A(\mathbf{x}) = \mathbf{0}$, then $\mathbf{x} = \mathbf{0}$.

PLEASE TURN OVER FOR MORE PROBLEMS

4. [Points: 30] In this question, $M = \begin{bmatrix} 1 & 3 & 3 & 0 \\ 2 & 6 & 6 & 8 \\ 1 & 3 & 3 & 4 \end{bmatrix}$.
   (a) Find the rank of M.
   (b) Find a basis for the null space, $\text{Nul}(M)$.
   (c) Find a basis for the column space, $\text{Col}(M)$.
   (d) Are the columns of M linearly independent? If not, find a linear dependence relation among them.

5. [Points: 5] Let L be a subspace of $\mathbb{R}^n$. Suppose $\mathbf{a} \in \mathbb{R}^n$ and let $K = \mathbf{a} + L$. Prove that if $\mathbf{v}$ and $\mathbf{w}$ are any two vectors in K, then $\mathbf{w} - \mathbf{v} \in L$.

6. [Points: 15] (Remember that $\mathbf{P}_n$ is a vector space.) In $\mathbf{P}_3$, define $T = \{x^3, 2x^2 - 1, 3x + 2\}$.
   (a) Is T linearly independent?
   (b) Does $x^3 - 5x$ belong to $\text{Span}(T)$?