Math 304-02 2006/5/2 **TEST III** Name ______________________

- Show all your work for each problem; show enough work to fully justify your answer.
- All numerical answers should be in terms of actual numbers.
- Start each numbered problem on a *fresh page.*
- Hand in *both* this paper and test booklet.

Remember that P_n is the set of polynomials in x of degree at most n. For use in some questions, here are some facts (don't prove them): Two ordered bases of $\mathbb{R}^3$ are

$$E = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \qquad \text{and} \qquad D = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right).$$

Two ordered bases of P_1 are

$$X = (x^0, x) \qquad \text{and} \qquad Z = (x+1, 2x-1).$$

1. A linear transformation $L : P_1 \to P_1$ has the properties

$$L(x+1) = x^0 \qquad \text{and} \qquad L(2x-1) = x.$$

   a. [Points: 7] Find the coordinate vector $K_Z(x)$.

   b. [Points: 7] Find $L(x)$.

   c. [Points: 7] Find the matrix ${}_XL_Z$.

2. A linear transformation $F : \mathbb{R}^3 \to P_1$ is defined by

$$F\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \right) = (a_3 - a_1)x + (a_3 + a_2).$$

   a. [Points: 7] Show that F is not an isomorphism.

   b. [Points: 7] Find the matrix ${}_XF_E$.

3. In this problem let

$$V = \{p(x) \in P_2 : p(2) = 0\}$$

   and

$$B = \{x - 2, x^2 - 2x\}.$$

   a. [Points: 4] What is the dimension $\dim P_2$? Justify by stating (without proof) a basis for P_2.

   b. [Points: 10] Prove that V is a subspace of P_2.

   c. [Points: 3] It is a fact that B is a basis of V (do *not* prove this). What is $\dim V$?

   d. [Points: 7] Which vector $\mathbf{v} \in P_2$ has coordinate vector $K_B(\mathbf{v}) = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$?

   e. [Points: 5] Use the Going Up Lemma to find a basis for P_2 that contains B.

4. [Points: 7] Define a function $G : \mathbb{R}^3 \to P_1$ by the formula

$$G(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}) = a_1 x + a_2 a_3.$$

Prove that G is *not* a linear transformation.

5. In this problem,
   - V is a vector space with basis B,
   - W is a vector space with basis C,
   - U is a vector space with basis A.

   We have two linear transformations,

$$F : V \to W \qquad \text{and} \qquad G : W \to U,$$

   whose matrices with respect to these bases are

$${}_C F_B = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & -1 \end{bmatrix} \qquad \text{and} \qquad {}_A G_C = \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 1 & -4 \end{bmatrix}.$$

   Then, GF is a linear transformation from V to U. (Don't prove this!)

   a. [Points: 7] Find the matrix ${}_A(GF)_B$ of GF with respect to B and A.

   b. [Points: 4] What are the dimensions $\dim V$ and $\dim W$?

   c. [Points: 4] Is F an isomorphism?

6. Use the bases E and D defined at the beginning of the test.

   a. [Points: 7] Find the matrix ${}_E I_D$ that changes from coordinates with respect to D to coordinates with respect to E in $\mathbb{R}^3$.

   b. [Points: 7] Explain briefly how to find the matrix ${}_D I_E$ from ${}_E I_D$. Do not calculate ${}_D I_E$.