Math 304 04 Test 2 (200 points)

You will have 90 minutes to complete this exam. Answer all questions in your blue book as completely as possible. Calculators and other computational devices are not allowed.

1. (8 points) What is the image of the unit square (i.e. the points $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$) under the linear transformation given by $\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$? Graph the image NEATLY in the plane.

2. (14 points) Let $\mathbf{A} : \mathbb{R}^5 \to \mathbb{R}^4$ be a rank 3 matrix.

a. What is the dimension of $ColSp(\mathbf{A})$?
b. What is the dimension of $RowSp(\mathbf{A})$?
c. What is the dimension of $NulSp(\mathbf{A})$?

3. (26 points) Consider the two collections of vectors in $\mathbb{R}^3$, $\mathbf{X} = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \right\}$ and $\mathbf{Y} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} \right\}$.

a. Show that $\mathbf{X}$ is a basis for $\mathbb{R}^3$.
b. Show that $\mathbf{Y}$ is a basis for $\mathbb{R}^3$.
c. Find the coordinates of the point $(2,1,3)$ with respect to each of these bases.
d. Find a change of basis matrix ${}_{\mathbf{X}}\mathbf{P}_{\mathbf{Y}}$ taking the basis $\mathbf{X}$ to the basis $\mathbf{Y}$.

4. (21 points) Let $\mathbf{M}$ be an $m \times n$ matrix regarded as a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Decide whether each of the following statements below are true or false. If the statement is false correct the underlined conclusion.

a. If the columns of $\mathbf{M}$ are linearly independent then $\underline{\mathbf{M} \text{ is onto}}$.
b. If the columns of $\mathbf{M}$ span $\mathbb{R}^m$ then $\underline{\mathbf{M} \text{ is one-to-one}}$.
c. If $\mathbf{M}$ is one-to-one then $\underline{NulSp(\mathbf{M}) = \{\mathbf{0}\}}$.
d. If $m < n$ then $\underline{NulSp(\mathbf{M}) \supsetneq \{\mathbf{0}\}}$.
e. If $n < m$ then $\underline{\mathbf{M} \text{ is not onto}}$.
f. If $rank(\mathbf{M}) = r$ then $\underline{\dim(NulSp(\mathbf{M})) = m - r}$.

5. (26 points) Let $\mathbf{X} = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix} \right\}$.

a. Show $\mathbf{X}$ is linearly dependent.

b. Using the "going down theorem" reduce $\mathbf{X}$ to a smaller, linearly independent collection $\mathbf{Y}$ with the same span.

c. Using your answer in part b. and the "going up theorem" to extend your $\mathbf{Y}$ to a basis of $\mathbb{R}^3$.

6. (25 points) Let $\mathbf{M} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 4 & 9 & 5 & 3 \\ 3 & 7 & 4 & 2 \\ -1 & 5 & 6 & -8 \end{bmatrix}$.

a. Find a basis for $ColSp(\mathbf{M})$. What is the dimension of $ColSp(\mathbf{M})$?

b. Find a basis for $NulSp(\mathbf{M})$. What is the dimension of $NulSp(\mathbf{M})$?

7. (12 points) Prove that if $\mathbf{X}$ is a collection of vectors in $\mathbb{R}^n$ then $Span(\mathbf{X})$ is a subspace of $\mathbb{R}^n$.

8. (12 points) Prove that if $\mathbf{X} = \{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k\}$ is a linearly independent collection of vectors in $\mathbb{R}^n$, then every vector in $Span(\mathbf{X})$ is expressed as a unique combination of the vectors of $\mathbf{X}$.

9. (30 points) Recall that $\{1, x, x^2\}$ is the standard basis for polynomial 3-space P_3, we give it this name because if we think of this set as actually $\{1 + 0x + 0x^2, 0 + 1x + 0x^2, 0 + 0x + 1x^2\}$ we get a natural isomorphism with $\mathbb{R}^3$ with basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

a. Using this natural isomorphism with $\mathbb{R}^3$, show that $\{1 + x, 2 + x + 2x^2, 4 + x^2\}$ is a basis for P_3 by finding the vectors in $\mathbb{R}^3$ that correspond to these polynomials and work in $\mathbb{R}^3$.

b. Given the vector $1 + x + x^2 \in P_3$ what are its coordinates with respect to the basis $\{1 + x, 2 + x + 2x^2, 4 + x^2\}$ from part a.?

10. (26 points) Consider the set $\mathbf{X} = \{\sin x, \cos x, e^{2x}\}$.as a basis for the vector space $V = Span(\mathbf{X})$. An arbitrary vector in V has the form $a \sin x + b \cos x + ce^{2x}$. Let $D : V \to V$ be the linear transformation of differentiation, that is, $D(a \sin x + b \cos x + ce^{2x}) = -b \sin x + a \cos x + 2ce^{2x}$.

a. $D(\sin x) = D(1 \sin x + 0 \cos x + 0e^{2x}) = ?$
b. $D(\cos x) = D(0 \sin x + 1 \cos x + 0e^{2x}) = ?$
c. $D(e^{2x}) = D(0 \sin x + 0 \cos x + 1e^{2x}) = ?$
d. Since every linear transformation is completely determined by the image of a basis, and V is 3 dimensional there exists a 3×3 matrix $\mathbf{M} : \mathbb{R}^3 \to \mathbb{R}^3$ that is isomorphic to the transformation $D : V \to V$. Find this $\mathbf{M}$.