Math 304 Test 3 (200 points)

Directions: Answer all questions as completely as possible in your blue book. Answers with no work receive no credit.

1. (15 points) Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$.

a. Calculate $\det(\mathbf{A})$ by using the definition and expanding along the first row.

b. Calculate $\det(\mathbf{A})$ by expanding along the second row.

c. Using your result determine if the matrix $\mathbf{A}$ invertible? Explain.

2. (15 points) Suppose the characteristic polynomial of a linear transformation is $p_T(\lambda) = \lambda^6 + 4\lambda^5 - 6\lambda^4 - 3\lambda^3 + 2\lambda^2 + \lambda$.

a. What is the dimension of the vector space in which the linear transformation acts?

b. Is T invertible? Explain.

3. (40 points) Let $\mathbf{M} = \begin{bmatrix} 2 & 2 & 4 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

a. Find the characteristic polynomial of $\mathbf{M}$.

b. Find the eigenvalues of $\mathbf{M}$.

c. Find a basis for each eigenspace of $\mathbf{M}$.

d. Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{M} = \mathbf{PDP}^{-1}$.

e. What are the eigenvalues of $\mathbf{M}^3$?

4. (18 points) Indicate whether each statement is true of false.

a. If $\mathbf{A}$ is singular then $\mathbf{A}$ is not diagonalizable.

b. If λ_1 is an eigenvalue of T then there exists a vector $\mathbf{v}$ such that $T\mathbf{v} = \lambda_1\mathbf{v}$.

c. If λ_2 is not an eigenvalue of T then the only vector that satisfies $T\mathbf{v} = \lambda_2\mathbf{v}$ is $\mathbf{v} = \mathbf{0}$.

d. $p_T(\lambda) = 3\lambda^4 - 2\lambda^3 + 7\lambda^2 + \lambda + 1$ is the characteristic polynomial of some linear transformation.

e. If $p_T(\lambda) = (2-\lambda)^5$ then the dimension of the $\lambda = 2$ eigenspace is 5.

f. Given any matrix $\mathbf{A}$ and any invertible matrix $\mathbf{B}$ the matrix $\mathbf{BAB}^{-1}$ has the same characteristic polynomial as $\mathbf{A}$.

5. (12 points) Let $\mathbf{u} = (1, 3, -2)$ and $\mathbf{v} = (4, 1, 1)$.

   a. Calculate $\mathbf{u} \cdot \mathbf{v}$. Are they orthogonal?
   b. Calculate the length (norm) of each vector.
   c. Normalize $\mathbf{u}$ so that it has length 1.

6. (12 points) Let $W = Span\left(\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\1\\2\end{bmatrix}\right\}\right)$. Find a basis for $W^\perp$.

7. (20 points) Let $\mathbf{u} = \begin{bmatrix}1\\2\\1\end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix}3\\1\\-5\end{bmatrix}$.

   a. Show $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
   b. Find the projection of $\begin{bmatrix}1\\1\\1\end{bmatrix}$ on to the subspace spanned by $\mathbf{u}$ and $\mathbf{v}$.

8. (20 points) Given the basis $\mathbf{X} = \left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\4\\3\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}\right\}$ apply the Gram-Schmidt process to this basis to form an orthogonal basis of $\mathbb{R}^3$.

9. (28 points) Consider the following inconsistent system of equations:
   $x_1 - x_2 = 2,$
   $2x_1 + x_2 = 2,$
   $x_1 + x_2 = 1.$

   a. Find the least squares solution of this system.
   b. Using your work in part a. find the projection of $\begin{bmatrix}2\\2\\1\end{bmatrix}$ on to the subspace spanned by $\begin{bmatrix}1\\2\\1\end{bmatrix}$ and $\begin{bmatrix}-1\\1\\1\end{bmatrix}$.

10. (20 points) Recall the definition of an inner product is a function from $V \times V \to \mathbb{R}$ that satisfies the following properties:

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$.
2. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$.
3. $\langle c\mathbf{u}, \mathbf{v} \rangle = c \langle \mathbf{u}, \mathbf{v} \rangle$.
4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$.
5. $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

Prove the dot product on $\mathbb{R}^n$ satisfies these 5 properties. (Hint: Write the vectors as $\mathbf{v} = (v_1, v_2, ..., v_n)$ and use the fact that $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^{n} u_i v_i$).