Do for discussion Mon., 5/1

An ordered basis for P_n is $X_n = (x^0, x, x^2, \ldots, x^n)$. An ordered basis for $\mathbb{R}^m$ is $E_m = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m)$.

Another ordered basis for $\mathbf{P}_2$ is $B = (x^2+1, x+1, 1)$. One ordered basis for $\mathbb{R}^3$ is $C = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ (the same as E_3). Another ordered basis for $\mathbb{R}^3$ is $D = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ where

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Still another ordered basis is $Y = (\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3)$ where

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

1. Define a function $L : \mathbf{P}_2 \to \mathbb{R}^3$ by

$$L(a_0x^0 + a_1x + a_2x^2) = \begin{bmatrix} a_1 + a_0 \\ a_1 + a_2 \\ a_0 + a_2 \end{bmatrix}.$$

   (a) Prove that L is a linear transformation.
   (b) Now assume that L is a linear transformation. Prove that L is an isomorphism by finding the inverse function $L^{-1} : \mathbb{R}^3 \to \mathbf{P}_2$.
   (c) Find

$$p_1(x) = L^{-1}(\mathbf{e}_1), \quad p_2(x) = L^{-1}(\mathbf{e}_2), \quad p_3(x) = L^{-1}(\mathbf{e}_3),$$

   and prove that $\{p_1(x), p_2(x), p_3(x)\}$ is a basis for $\mathbf{P}_2$.

2. (a) Find the matrix ${}_CL_B$ of L with respect to ordered bases B and C.
   (b) Find the matrix ${}_DL_B$ of L with respect to B and D.
   (c) Is L an isomorphism? (Solve by using part (a) or (b), and not the method of the preceding problem.)

3. A linear transformation $F : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$F(\mathbf{w}_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad F(\mathbf{w}_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad F(\mathbf{w}_3) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

   (a) Find the matrix ${}_CF_D$.
   (b) Find the matrix ${}_D(F^{-1})_C$. Multiply by ${}_CF_D$: what do you get?
   (c) Find the matrix ${}_CF_C$ and compare with (a).
   (d) Find ${}_C(F^{-1})_C$ and compare with (b). Multiply by ${}_CF_C$: what do you get? What does that say about the relationship between these two matrices?
   (e) Find ${}_DF_C$ and compare with (a), (b), and (c).

4. A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by having matrix

$$_{E_2}T_D = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 1 \end{bmatrix}.$$

Find a formula for

$$T(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}).$$

5. A linear transformation $G : V \to W$ has the matrix

$$\begin{bmatrix} 4 & 8 & 0 & -1 \\ 0 & 0 & 3 & 1 \\ -2 & 0 & -3 & 1 \end{bmatrix}$$

with respect to bases X and Y.

(a) What are the dimensions of V and W?

(b) If V is a P_n and W is an $\mathbb{R}^m$, what are n and m?

(c) Use the ordered bases $X = X_n$ for $\mathbf{P}_n$ and $Y = E_m$ for $\mathbb{R}^m$. With your values for n and m from part (b), write out X and Y completely.

(d) Find a direct formula for G of the form:

$$G(a_0 + a_1 x + \cdots + a_n x^n) = \text{ (fill in the correct matrix here).}$$

6. Here we just have one vector space, $\mathbb{R}^3$, but different bases.

(a) Find the basis-change matrix $_Y I_D$ that changes coordinates with respect to D in $\mathbb{R}^3$ to coordinates with respect to Y.

(b) Find the basis-change matrix $_D I_Y$ that changes from coordinates with respect to Y to coordinates with respect to D. Multiply the answers to (a) and (b): what do you get? Is that a coincidence or will it always happen that way?

(c) Find the basis-change matrix $_C I_D$ that changes coordinates with respect to D in $\mathbb{R}^3$ to coordinates with respect to C.

(d) Find the basis-change matrix $_D I_C$ that changes from coordinates with respect to C to coordinates with respect to D. Multiply the answers to (c) and (d): what do you get? Does this suggest an easier way to find the answer to part (d)?