Math 304-02 (Zaslavsky) **Problem Set VIII** Fri., 3/24/2006

Do for discussion Mon., 3/27/2006

In these problems, $A = \begin{bmatrix} 2 & 10 & 0 \\ 1 & 5 & 6 \\ -1 & -5 & 6 \end{bmatrix}$.

Also, $\mathbf{P}_n$ is the set of all polynomials whose degree is no greater than n.

1. Let L be a subspace of $\mathbb{R}^n$. Suppose $\mathbf{a} \in \mathbb{R}^n$ and let $L' = \mathbf{a} + L$.
   (a) Prove that if $\mathbf{v}$ and $\mathbf{w}$ are any two vectors in L', then $\mathbf{w} - \mathbf{v} \in L$.
   (b) Prove that, if $\mathbf{v}$ is any vector in L' and $\mathbf{x}$ is any vector in L, then $\mathbf{v} + \mathbf{x} \in L'$.

2. Let K be an affine set in $\mathbb{R}^n$ and let $\mathbf{a} \in K$. Prove that the set
$$-\mathbf{a} + K = \{-\mathbf{a} + \mathbf{v} : \mathbf{v} \in K\}$$
is a subspace of $\mathbb{R}^n$.

3. Find all solutions of the linear system $A\mathbf{x} = \mathbf{0}$. But first, answer the question: for which number n is every solution an element of $\mathbb{R}^n$?

4. Find all solutions of the linear system $A\mathbf{x} = \begin{bmatrix} -4 \\ -2 \\ 4 \end{bmatrix}$. Use your solution of problem 3 to solve this problem. Express your answer in parametric form.

5. (a) Take two different solutions in problem 4 and calculate their sum. Is it in the solution set of $A\mathbf{x} = \mathbf{0}$? Is it in the null space of A?
   (b) Now do the same for the difference of the two vectors.
   (c) Now do the same for the zero vector.

6. Find a basis for $\mathrm{Nul}(A)$, also for $\mathrm{Col}(A)$, and for $\mathrm{Row}(A)$.

7. Find a vector $\mathbf{a}$ such that the affine set $\mathbf{a} + \mathrm{Nul}(A)$
   (a) is,
   (b) is not
   a subspace of $\mathbb{R}^3$.

8. What are the dimensions of $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$? What is the sum of the dimensions? How can you explain this sum?

9. (a) Find a basis for the vector space $\mathbf{P}_2$.
   (b) Prove it is a basis.
   (c) What is the dimension of $\mathbf{P}_2$?
   (d) What is the zero vector, $\mathbf{0}$, in $\mathbf{P}_2$?

10. In $\mathbf{P}_2$, (a) prove that $\mathrm{Span}\{x^2, 2x^2 - 1, 3x + 2\}$ is a subspace, and (b) find a basis for it. (c) What is its dimension?

11. In $\mathbf{P}_3$, take the subset $S = \{x^3 + x + x^0, 2x^3 - x^0, 3x + 2x^0, -x^3 + x\}$.
   (a) Prove that $\text{Span}(S)$ is a subspace.
   (b) Find a basis for $\text{Span}(S)$ that is a subset of S.
   (c) Find a basis for $\text{Span}(S)$ that is not a subset of S.
   (d) What is the dimension of $\text{Span}(S)$?
   (e) Find a vector in $\mathbf{P}_3$ that is not in $\text{Span}(S)$.

12. Define a function $F : \mathbf{P}_3 \to \mathbf{P}_2$ by

$$F(a_3x^3 + a_2x^2 + a_1x + a_0x^0) = a_0x^2 + (a_0 - 2a_2)x + (a_1 + a_3)x^0.$$

   (a) Show that F is a linear transformation.
   (b) Find $\text{Ker}(F)$. Find a basis for $\text{Ker}(F)$. What is $\dim \text{Ker}(F)$?
   (c) Find $\text{Image}(F)$. Find a basis for $\text{Image}(F)$. What is $\dim \text{Image}(F)$?
   (d) Compare $\dim \text{Ker}(F) + \dim \text{Image}(F)$ with $\dim \mathbf{P}_3$.