Do for discussion Tues., 4/4

$\mathbf{P}$ is the set of all polynomials in x, and $\mathbf{P}_n$ is the subset consisting of all polynomials of degree at most n.

1. Prove **Lemma A**: In any vector space V, we have $0\mathbf{x} = \mathbf{0}$ for every $\mathbf{x} \in V$. Use the 8 properties of a vector space (page 42).

2. Prove **Lemma B**: In any vector space V, the negative of a vector $\mathbf{x}$ is given by $-\mathbf{x} = (-1)\mathbf{x}$. Use the 8 properties, and also you may find you can use Lemma A.

3. (a) Let $V^0 = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^3 \text{ and } x_1 + x_2 + x_3 = 0\}$. Show that V^0 is a subspace of $\mathbb{R}^3$. Deduce that V^0 is a vector space.
(b) Let $V^3 = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^3 \text{ and } x_1 + x_2 + x_3 = 3\}$. Show that V^3 is not a subspace of $\mathbb{R}^3$ and is not a vector space.

4. (a) Show that the set $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ spans $\mathbb{R}^3$.
(b) Find a linear dependence among the members of S.
(c) Use your linear dependence to express one member of S as a linear combination of the others.
(d) Let T be S with the member in part (c) removed. Show that T spans $\mathbb{R}^3$. Preferably, use the answers to (a) and (c) to shorten your work.
Notice that (b)–(d) are the steps in using the "Going Down Lemma" 5.2.3.

5. Use the "Going Down Lemma" 5.2.3 to find a basis for $\mathbb{R}^3$ that is a subset of S in Question 4. (To do this, use the method of Question 4 as often as necessary to make S smaller until it becomes linearly independent.)

6. (a) Prove that the set $T = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$ spans V^0 of Exercise 3 and is linearly independent.
(b) Show that T does not span $\mathbb{R}^3$.
(c) Use the "Going Up Lemma" 5.2.2 to find a new vector in $\mathbb{R}^3$, $\mathbf{u}$, such that $R = T \cup \{\mathbf{u}\}$ is linearly independent. Show that R is a basis for $\mathbb{R}^3$.

7. (a) Let $\mathbf{P}^0 = \{p(x) : p(x) \in \mathbf{P} \text{ and } p(1) = 0\}$. Show that $\mathbf{P}^0$ is a subspace of $\mathbf{P}$. Deduce that $\mathbf{P}^0$ is a vector space.
(b) Let $\mathbf{P}^3 = \{p(x) : p(x) \in \mathbf{P} \text{ and } p(1) = 3\}$. Show that $\mathbf{P}^3$ is not a subspace of $\mathbf{P}$ and is not a vector space.
(c) Let $\mathbf{P}_3^0 = \{p(x) : p(x) \in \mathbf{P}_3 \text{ and } p(1) = 0\}$. Show that $\mathbf{P}_3^0$ is a subspace of $\mathbf{P}_3$. Deduce that $\mathbf{P}_3^0$ is a vector space.

8. (a) Show that the set $S = \{x^3 - 4x, 2x^3 + x^2 - 2, -x^2 - 3x + 1, x^2 - 4, x^2, 3x\}$ spans $\mathbf{P}_3$.
(b) Find a linear dependence among the members of S.
(c) Use your linear dependence to express one member of S as a linear combination of the others.
(d) Let T be S with the member in part (c) removed. Show that T spans $\mathbf{P}_3$. Preferably, use the answers to (a) and (c) to shorten your work.
Notice that (b)–(d) are the steps in using the "Going Down Lemma" 5.2.3.

9. Use the "Going Down Lemma" 5.2.3 to find a basis for $\mathbf{P}_3$ that is a subset of S in Question 8. (To do this, use the method of the Question 8 as often as necessary to make S smaller until it becomes linearly independent.)

10. (a) Prove that the set $S = \{x^0, x^1, x^2, \ldots\} \subseteq \mathbf{P}$ spans $\mathbf{P}$ and is linearly independent.
(b) Show that S does not span $\mathbf{P}_4$.
(c) Use the "Going Up Lemma" 5.2.2 to find a new vector (i.e., polynomial) in $\mathbf{P}_4$, $\mathbf{u}$, such that $R = S \cup \{\mathbf{u}\}$ is linearly independent . Show that R is a basis for $\mathbf{P}_4$.

11. (a) Prove that the set $T = \{x^1 - x^0, x^2 - x^0, x^3 - x^0\}$ spans $\mathbf{P}_3^0$ and is linearly independent.
(b) Show that T does not span $\mathbf{P}_3$.
(c) Use the "Going Up Lemma" 5.2.2 to find a new vector (i.e., polynomial) in $\mathbf{P}_3$, $\mathbf{u}$, such that $R = T \cup \{\mathbf{u}\}$ is linearly independent subset. Show that R is a basis for $\mathbf{P}_3$.