Do for discussion Tues., 4/4

$\mathbf{P}$ is the set of all polynomials in x, and $\mathbf{P}_n$ is the subset consisting of all polynomials of degree at most n.

1. Prove **Lemma A**: In any vector space V, we have $0\mathbf{x} = \mathbf{0}$ for every $\mathbf{x} \in V$. Use the 8 properties of a vector space (page 42).

   Solution. Consider the sum $(0+0)\mathbf{x}$. Since $0+0=0$ (ordinary arithmetic with real numbers),
$$(0+0)\mathbf{x} = 0\mathbf{x}.$$
   By one of the distributive laws,
$$(0+0)\mathbf{x} = 0\mathbf{x} + 0\mathbf{x}.$$
   Therefore,
$$0\mathbf{x} + 0\mathbf{x} = 0\mathbf{x}.$$
   We know $0\mathbf{x}$ has a negative, $-0\mathbf{x}$. We add this to both sides:
$$(0\mathbf{x} + 0\mathbf{x}) + (-0\mathbf{x}) = 0\mathbf{x} + (-0\mathbf{x}).$$
   Then we use the associative law of addition to get
$$0\mathbf{x} + \big(0\mathbf{x} + (-0\mathbf{x})\big) = 0\mathbf{x} + (-0\mathbf{x}).$$
   Next, we use the definition of a negative to simplify this to
$$0\mathbf{x} + \mathbf{0} = \mathbf{0}.$$
   Finally, we use the definition of a zero vector to simplify this to
$$0\mathbf{x} = \mathbf{0}.$$

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2. Prove **Lemma B**: In any vector space V, the negative of a vector $\mathbf{x}$ is given by $-\mathbf{x} = (-1)\mathbf{x}$. Use the 8 properties, and also you may find you can use Lemma A.

   Solution. To show a vector $\mathbf{y}$ is the negative of a vector $\mathbf{x}$, we show that $\mathbf{x}+\mathbf{y} = \mathbf{0}$. Let's prove that $\mathbf{y} = (-1)\mathbf{x}$ has this property.
$$\begin{aligned} \mathbf{x} + (-1)\mathbf{x} &= 1\mathbf{x} + (-1)\mathbf{x} && \text{by Property 8, page 42} \\ &= \big(1 + (-1)\big)\mathbf{x} && \text{by the distributive law} \\ &= 0\mathbf{x} = \mathbf{0} && \text{by Lemma A.} \end{aligned}$$

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3. (a) Let $V^0 = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^3 \text{ and } x_1 + x_2 + x_3 = 0\}$. Show that V^0 is a subspace of $\mathbb{R}^3$. Deduce that V^0 is a vector space.

Solution. We have to prove
(1) $\mathbf{u}, \mathbf{v} \in V^0$ imply that $\mathbf{u} + \mathbf{v} \in V^0$,
(2) $\mathbf{u} \in V^0$, $c \in \mathbb{R}$ imply that $c\mathbf{u} \in V^0$, and
(3) $\mathbf{0} \in V^0$.

Proof of (1): We know that $u_1 + u_2 + u_3 = 0$ and $v_1 + v_2 + v_3 = 0$. Adding, $(u_1 + v_1) + (u_2 + v_2) + (u_3 + v_3) = 0$ (*). Since $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$, equation (*) tells us that $\mathbf{u} + \mathbf{v} \in V^0$.

Proof of (2): We know that $u_1 + u_2 + u_3 = 0$. Multiplying by c, we get $cu_1 + cu_2 + cu_3 = 0$ (**). Since $c\mathbf{u} = (cu_1, cu_2, cu_3)$, equation (**) tells us that $c\mathbf{u} \in V^0$.

Proof of (3): $\mathbf{0} = (0, 0, 0)$, and the sum of the entries is 0, so $\mathbf{0} \in V^0$.

(b) Let $V^3 = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^3 \text{ and } x_1 + x_2 + x_3 = 3\}$. Show that V^3 is not a subspace of $\mathbb{R}^3$ and is not a vector space.

Solution. There are several ways to prove this, since V^3 violates all three properties of a vector space. For instance, the fact that $\mathbf{0} \notin V^3$ (because $0 + 0 + 0 \neq 3$) shows that V^3 violates property (3) mentioned in part (a).

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4. (a) Show that the set $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ spans $\mathbb{R}^3$.

Solution. We can use known matrix methods, since $\text{Span}(S) = \text{Col}(M)$ if we form the matrix $M = \begin{bmatrix} 1 & 1 & 0 & 2 & -2 \\ 0 & 1 & 1 & -1 & 0 \\ 1 & 3 & 0 & 0 & 1 \end{bmatrix}$. Find the pivot positions of M by finding a row echelon form. There is a pivot position in every row. (The pivot columns are the first, second, and third columns.) Therefore, $\text{Col}(M)$ equals all of $\mathbb{R}^3$; that is, S spans all of $\mathbb{R}^3$.

(b) Find a linear dependence among the members of S.

Solution. You might be able to guess a linear dependence (then you'll have to check it—show complete work!), but it's better to know a general method to find one, or prove none exists. The general method is to find the null space of M. If the null space is the zero vector space, there are no linear dependencies. Any nonzero element of the null space gives a linear dependence relation.

We find the null space in the usual way, by reducing M to reduced row echelon form, which is $\begin{bmatrix} 1 & 0 & 0 & 3 & -\frac{7}{2} \\ 0 & 1 & 0 & -1 & \frac{3}{2} \\ 0 & 0 & 1 & 0 & -\frac{3}{2} \end{bmatrix}$. Thus the general solution to $M\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = x_4 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} \frac{7}{2} \\ -\frac{3}{2} \\ \frac{3}{2} \\ 0 \\ 1 \end{bmatrix}, \quad \text{i.e., } \mathrm{Nul}(M) = \mathrm{Span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{7}{2} \\ -\frac{3}{2} \\ \frac{3}{2} \\ 0 \\ 1 \end{bmatrix} \right\}.$$

These vectors (for all choices of real numbers x_4, x_5) are all the elements of the null space. Pick any one that isn't $\mathbf{0}$ and you have a linear dependence. For instance, pick $(-3, 1, 0, 1, 0)$ and you have the dependence

(A) $$(-3) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0}.$$

(c) Use your linear dependence to express one member of S as a linear combination of the others.

Solution. In equation (A) we can solve for any of the three vectors whose coefficients are nonzero. I'll pick the first: then

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}.$$

(That's my first solution.)

But if I picked the last vector, then I'd get

$$\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}.$$

There are other possible solutions, too. Do you see them?

(d) Let T be S with the member in part (c) removed. Show that T spans $\mathbb{R}^3$. Preferably, use the answers to (a) and (c) to shorten your work.

Solution. In my first solution to (c), $T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$. I'm stuck: I can't use (a) and (c) to help me. I'll have to find the pivot positions of the matrix formed by these four vectors, which means work. I'll omit this work because it's similar to the work in (c).

But if I picked the last vector, as in my second solution to (c), then

$$T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

The matrix with these vectors as columns is $M' = \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{bmatrix}$. You'll notice that reducing this to row echelon form uses exactly the same operations as with M. That's because the column I removed is a nonpivot columns, so the pivot columns are the same as in M, and it's the pivot columns that tell you which row operations to do. Since we do the same row operations on the same columns (except one), we'll get the same pivot columns. Therefore, the reduced row echelon form of M' can be obtained from that of M by We'll have 3 pivot columns for the 3 rows, which means that $\mathrm{Col}(M') = \mathbb{R}^3$. That means T spans $\mathbb{R}^3$. (See, I didn't have to do any calculations!)

Notice that (b)–(d) are the steps in using the "Going Down Lemma" 5.2.3.

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5. Use the "Going Down Lemma" 5.2.3 to find a basis for $\mathbb{R}^3$ that is a subset of S in Question 4. (To do this, use the method of Question 4 as often as necessary to make S smaller until it becomes linearly independent.)

Solution. The first step is to find one vector to remove from S. We did this in Question 4, getting T (which is S with one vector removed). We did it by picking a vector that is a linear combination of the other vectors. We don't have to verify that the new set spans the same subspace as the old set, because the "Going Down Lemma" guarantees that will be true.

The second step is to find a vector in T that is a linear combination of the other vectors in T. Of course, which one it is depends on which T you picked in 4(c, d). I'll follow the second solution in Question 4, where I got

$$T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

The matrix of these vectors, M' (from 4(d)), has reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -\frac{7}{2} \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{3}{2} \end{bmatrix}$.

Thus,

$$\mathrm{Nul}(M') = \mathrm{Span}\left\{ \begin{bmatrix} \frac{7}{2} \\ -\frac{3}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix} \right\}.$$

A linear dependence is

$$\frac{7}{2}\begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{3}{2}\begin{bmatrix}1\\1\\3\end{bmatrix} + \frac{3}{2}\begin{bmatrix}0\\1\\0\end{bmatrix} + \begin{bmatrix}-2\\0\\1\end{bmatrix} = \mathbf{0}.$$

You can solve for any one of these four vectors, so any one of them is a linear combination of the other three. You can delete any one from T to get a new set U that has the same span. For instance, I will delete the last vector, giving

$$U = \left\{ \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\3\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}.$$

If U is linearly dependent, we have to keep going like this, removing one vector at a time. But if U is linearly independent, we have a basis for $\mathrm{Span}(U)$ (which $= \mathrm{Span}(T) = \mathrm{Span}(S) = \mathbb{R}^3$). We check U for linear dependence in the usual way: we make a matrix M'' from the three vectors, and find the pivot positions. It turns out that every column has a pivot position. Thus, U is independent. Therefore, U is a basis for $\mathrm{Span}(U) = \mathbb{R}^3$. We're done!

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6. (a) Prove that the set $T = \left\{ \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\-2\\1\end{bmatrix} \right\}$ spans V^0 of Exercise 3 and is linearly independent.

Solution. First we show that $T \subseteq V^0$. $(1,0,-1) \in V^0$ because the sum of its entries is 0. Similarly, $(1,-2,1) \in V^0$ because the sum of its entries is 0.

Now we show that every $\mathbf{x} \in V^0$ is a linear combination of the vectors in T. We do this by showing that the equation

$$c_1\begin{bmatrix}1\\0\\-1\end{bmatrix} + c_2\begin{bmatrix}1\\-2\\1\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

has a solution for every $\mathbf{x}$ such that $x_1 + x_2 + x_3 = 0$. Set up the augmented matrix and row reduce:

$$\begin{bmatrix}1 & 1 & x_1\\0 & -2 & x_2\\-1 & 1 & x_3\end{bmatrix} \to \begin{bmatrix}1 & 1 & x_1\\0 & -2 & x_2\\0 & 2 & x_1+x_3\end{bmatrix} \to \begin{bmatrix}1 & 1 & x_1\\0 & -2 & x_2\\0 & 0 & x_1+x_2+x_3\end{bmatrix}$$

The original equation is solvable $\iff$ if there is no pivot position in the last column $\iff x_1+x_2+x_3 = 0 \iff \mathbf{x} \in V^0$. Thus, we conclude that $\mathbf{x} \in \mathrm{Span}(T) \iff \mathbf{x} \in V^0$. Stated in a different way: $\mathrm{Span}(T) = V^0$.

(b) Show that T does not span $\mathbb{R}^3$.

Solution. Since we found the exact span of T in (a), namely V^0, and $V^0 \neq \mathbb{R}^3$ obviously, we also proved that T does not span $\mathbb{R}^3$.

(c) Use the "Going Up Lemma" 5.2.2 to find a new vector in $\mathbb{R}^3$, $\mathbf{u}$, such that $R = T \cup \{\mathbf{u}\}$ is linearly independent. Show that R is a basis for $\mathbb{R}^3$.

Solution. According to the "Going Up Lemma", we need to find $\mathbf{u} \notin V^0$. This is easy. I will pick $\mathbf{u} = (0,0,1)$, which is not in V^0 because the sum of its entries is not 0.

Now we have to show that $R = T \cup \{\mathbf{u}\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$. We form the matrix $M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ -1 & 1 & 1 \end{bmatrix}$ and row reduce to find the rank. (As usual in this solutions handout, I omit the details, which *you* should *always* include when you hand in your solutions to me!) We'll find that the rank of M is 3, which equals the number of rows; therefore (by some theorem in Chapter 4), R spans $\mathbb{R}^3$. We already knew R was independent by the "Going Up Lemma", but another way we know it is by the fact that the rank of M equals the number of columns. Either way, since R is independent and spans $\mathbb{R}^3$, it is a basis for $\mathbb{R}^3$.

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7. (a) Let $\mathbf{P}^0 = \{p(x) : p(x) \in \mathbf{P} \text{ and } p(1) = 0\}$. Show that $\mathbf{P}^0$ is a subspace of $\mathbf{P}$. Deduce that $\mathbf{P}^0$ is a vector space.

Solution. This is similar to Question 3(a). I will post details when I have more time, but the three properties to prove are the same.

(b) Let $\mathbf{P}^3 = \{p(x) : p(x) \in \mathbf{P} \text{ and } p(1) = 3\}$. Show that $\mathbf{P}^3$ is not a subspace of $\mathbf{P}$ and is not a vector space.

Solution. This is similar to Question 3(b). For instance, the zero vector in $\mathbf{P}$ is the zero polynomial, $p(x) = 0$, and it is not in $\mathbf{P}^3$ because evaluating it at $x = 0$ gives 0, not 3.

(c) Let $\mathbf{P}_3^0 = \{p(x) : p(x) \in \mathbf{P}_3 \text{ and } p(1) = 0\}$. Show that $\mathbf{P}_3^0$ is a subspace of $\mathbf{P}_3$. Deduce that $\mathbf{P}_3^0$ is a vector space.

Solution. The solution will be exactly like the solution of part (a).

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8. (a) Show that the set $S = \{x^3 - 4x, 2x^3 + x^2 - 2, -x^2 - 3x + 1, x^2 - 4, x^2, 3x\}$ spans $\mathbf{P}_3$.

Solution. We have to show that, for any $p(x) \in \mathbf{P}_3$, $p(x)$ is a linear combination of S. That means: for any polynomial $a_3x^3 + a_2x^2 + a_1x + a_0$, the equation

$$\begin{aligned} c_1(x^3 - 4x) + &c_2(2x^3 + x^2 - 2) + c_3(-x^2 - 3x + 1) + c_4(x^2 - 4) + c_5(x^2) + c_6(3x) \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 \end{aligned} \tag{P}$$

has a solution $\mathbf{c} = (c_1, c_2, c_3, c_4, c_5, c_6)$. The method is to collect terms in Equation (P) and compare coefficients of like powers on both sides; this gives you a linear system in which the unknowns are $c_1, c_2, \ldots, c_6$. You have to show there is a solution to this system, no matter what the real numbers $a_0, \ldots, a_3$ happen to be. You do this by the usual matrix method.

(b) Find a linear dependence among the members of S.

Solution. You do this by finding a nontrivial solution (that is, not all c_i equal to 0) for Equation (P) with right-hand side 0. Otherwise, it's similar to Question 4(b).

(c) Use your linear dependence to express one member of S as a linear combination of the others.

Solution. Similar to Question 4(c).

(d) Let T be S with the member in part (c) removed. Show that T spans $\mathbf{P}_3$. Preferably, use the answers to (a) and (c) to shorten your work.

Solution. Similar to Question 4(d).

Notice that (b)–(d) are the steps in using the "Going Down Lemma" 5.2.3.

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9. Use the "Going Down Lemma" 5.2.3 to find a basis for $\mathbf{P}_3$ that is a subset of S in Question 8. (To do this, use the method of Question 8 as often as necessary to make S smaller until it becomes linearly independent.)

   Solution. Similar to Question 4. I may post a solution if I have time.

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10. (a) Prove that the set $S = \{x^0, x^1, x^2, \ldots\} \subseteq \mathbf{P}$ spans $\mathbf{P}$ and is linearly independent.

    Solution. The proof that S is independent was given in class and appears in the book. The proof that it spans $\mathbf{P}$ is simply that every polynomial has the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 x^0$, which is a linear combination of the monomials $x^n, x^{n-1}, \ldots, x, x^0$, which are elements of S.

    (b) Show that S does not span $\mathbf{P}_4$.

    Solution. Span(S) contains x^5, which is not an element of the set $\mathbf{P}_4$. Therefore, S does not span $\mathbf{P}_4$.

    (c) Use the "Going Up Lemma" 5.2.2 to find a new vector (i.e., polynomial) in $\mathbf{P}_4$, $\mathbf{u}$, such that $R = S \cup \{\mathbf{u}\}$ is linearly independent. Show that R is a basis for $\mathbf{P}_4$.

    Solution.

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11. (a) Prove that the set $T = \{x^1 - x^0, x^2 - x^0, x^3 - x^0\}$ spans $\mathbf{P}_3^0$ and is linearly independent.

Solution. Similar to Question 6(a), using the idea of the setup of Question 8 as in Equation (P).

(b) Show that T does not span $\mathbf{P}_3$.

Solution.

(c) Use the "Going Up Lemma" 5.2.2 to find a new vector (i.e., polynomial) in $\mathbf{P}_3$, $\mathbf{u}$, such that $R = T \cup \{\mathbf{u}\}$ is linearly independent subset. Show that R is a basis for $\mathbf{P}_3$.

Solution.