List of Quizzes for
Math 304-07, Linear Algebra, Spring 2017
Prof. Zaslavsky

Quiz 1, 1/20.

1. What does it mean to say a linear system is consistent?
   Answer: It means the system has a solution.
2. Given a linear system, how do you use matrices to find out whether it is consistent?
   Answer: You reduce the augmented matrix to row echelon form so you can find the pivot positions. If there is a pivot position in the constants column (the right-hand column), the system is inconsistent. If there is not, it is consistent.

Quiz 2, 1/25.

1. Write out without a summation sign: $\displaystyle\sum_{i=3}^{8} i$.
   Answer: $3+4+5+6+7+8$
2. The same, for $\displaystyle\sum_{i=3}^{n} i$, where n is an integer ≥ 3.
   Answer: $3+4+\cdots+n$
3. (Bonus) Do you know the formula for $\displaystyle\sum_{i=1}^{n} i$? If so, write it down.
   Answer: $n(n+1)/2$.

Quiz 3, 1/27.

Let $C = \begin{bmatrix} 3 & 1 & 3 \\ 2 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$. Use C to make a function $F_C(\mathbf{x})$ (or $C(\mathbf{x})$ in the book's notation).

Then answer:

1. What is the domain of F_C? Answer $\underline{\quad \mathbb{R}^3 \quad}$
   Answer: The answer is not a number. It is a set.
2. What is the codomain of F_C? Answer $\underline{\quad \mathbb{R}^4 \quad}$
3. Evaluate $F_C(\mathbf{x})$ when $\mathbf{x} = (1,2,3)$. Answer $\underline{\quad (14,1,4,1) \quad}$
   Answer: The answer is not a number. It is a vector that belongs to $\mathbb{R}^4$, i.e., it has 4 components.

Quiz 4, 2/1.

Matrices: $A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 1 \end{bmatrix}$, $\quad B = \begin{bmatrix} 3 & 1 \\ 4 & 2 \\ 5 & 3 \end{bmatrix}$

1. How do you start an e-mail to your linear algebra teacher?
   Answer: “Dear Prof. Zaslavsky,” or “Hello, Prof. Zaslavsky,” or “Good morning, Prof. Zaslavsky,” etc. Avoid the use of “Hey”.
2. Find the product AB.
   Answer: The product is the 2×2 matrix $\begin{bmatrix} 18 & 10 \\ 16 & 8 \end{bmatrix}$. Some multiplied BA, which doesn’t have the right size.

Quiz 5, 2/3.

Let $C = \begin{bmatrix} 2 & 4 & 4 \\ 2 & 1 & 0 \end{bmatrix}$.

1. Is the function F_C one-to-one?
   Answer: No, because [there are several good reasons, but you *must* give a reason] after you find the pivot positions, you see (1) the rank is $r = 2$ but the number of columns is $n = 3$ so $r \neq n$; or (2) there is a free variable. You cannot give an answer that isn’t a guess if you don’t reduce C.
2. Is the function F_C onto?
   Answer: Yes, because $r = 2$ and the number of rows is $m = 2$, so $r = m$. (Some people got m and n mixed up. Better: remember which question depends on the number of rows and which question depends on the number of columns.)
3. What is the image of F_C?
   Answer: $\mathbb{R}^2$, because the function is onto. (The image of a function cannot be a number or a vector. It has to be a set.)

Quiz 6, 2/6.

1. Define a matrix.
   Answer: A matrix is a rectangular array of numbers (which may be variables or expressions, but they are not sets or vectors).
2. Let $C = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \end{bmatrix}$. Is F_C
   (a) onto,
   (b) one-to-one
   (with reasons), and:
   (c) What is the image of F_C?

Answer: When reduced to row echelon form, C becomes $C' = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Thus, it has rank 2, which is less than the number of rows (so it is not onto) and the number of columns (so it is not one-to-one).

To answer (c) you need to reduce an augmented matrix, $\begin{bmatrix} 1 & 2 & 3 & y_1 \\ 1 & 3 & 4 & y_2 \\ 1 & 4 & 5 & y_3 \end{bmatrix}$, which reduces to $\begin{bmatrix} 1 & 2 & 3 & y_1 \\ 0 & 1 & 1 & y_2 - y_1 \\ 0 & 0 & 0 & y_3 - 2y_2 + y_1 \end{bmatrix}$. To find the image the only question is whether there is a pivot position in the last column. There is, if $y_3 - 2y_2 + y_1 \neq 0$; then the vector (y_1, y_2, y_3) is not a value of the function F_C, so by the definition of image, it is not in the image. But if $y_3 - 2y_2 + y_1 = 0$, there is no pivot position in the last column; then (y_1, y_2, y_3) is a value of the function F_C, so by the definition of image, it is in the image. Therefore, the image is the set $\{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_3 - 2y_2 + y_1 = 0\}$, which can be written more simply since we don't need three variables; thus the image is

$$\{(y_1, y_2, 2y_2 - y_1) \in \mathbb{R}^3 : y_1, y_2 \in \mathbb{R}\}.$$

Note that this is a *set*, and it is a set of *vectors* in the codomain.

Quiz 7, 2/10.

Let $C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \end{bmatrix}$. Find the image of the linear transformation $F_C : \mathbb{R}^3 \to \mathbb{R}^2$.

Answer: The rank of this matrix is 2, which equals the number of rows, so F_C is onto; that means the image is $\mathbb{R}^2$, the entire codomain.

Note that a matrix does not have an image and neither a matrix nor a function can have a free variable! A *function* has an image, and a *linear system* can have a free variable.

Quiz 8, 2/16.

Find the matrix for reflecting $\mathbb{R}^2$ in the x_2-axis. [This is (21)4 from the book.]

Answer: $\mathbf{e}_1 \mapsto -\mathbf{e}_1$ and $\mathbf{e}_2 \mapsto \mathbf{e}_2$, so the matrix is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

Quiz 9, 2/16.

Find the image of $F_A : \mathbb{R}^3 \to \mathbb{R}^3$ where $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Answer: Method 1. Think of the value $F_A(\mathbf{x}) = \mathbf{y} = (y_1, y_2, y_3)$ where $\mathbf{y} = A\mathbf{x}$. That means we have a linear system

$$\begin{aligned} 3x_1 + 2x_2 + 1x_3 &= y_1, \\ 4x_1 + 3x_2 + 1x_3 &= y_2, \\ 1x_1 + 1x_2 + 0x_3 &= y_3. \end{aligned}$$

We know $\mathbf{y}$ is in the image of F_A if and only if this system has a solution. So, we row reduce:

$$\big[A \mid \mathbf{y}\big] = \begin{bmatrix} 3 & 2 & 1 & y_1 \\ 4 & 3 & 1 & y_2 \\ 1 & 1 & 0 & y_3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & y_3 \\ 0 & -1 & 1 & y_1 - 3y_3 \\ 0 & 0 & 0 & y_2 - y_1 - y_3 \end{bmatrix}.$$

The last row tells us that the linear system is solvable (that is, $\mathbf{x}$ exists) if and only if $y_1 - y_2 + y_3 = 0$. Therefore, the image is

$$\mathrm{Image}(F_A) = \{(y_1, y_2, y_3) \mid (y_1, y_2, y_3) \in \mathbb{R}^3,\ y_1 - y_2 + y_3 = 0\}.$$

This means that the vectors with $y_1 - y_2 + y_3 = 0$ are all the vectors $\mathbf{y}$ that can be obtained by applying the linear transformation F_A.

Quiz 10, 3/2.

Let $X = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 8 \\ 16 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$. I will call these vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ for short.

1. Is X linearly independent? If not, write one of the vectors $\mathbf{v}_i$ as a linear combination of the others.

   Answer: (a) X is linearly independent if and only if the vector equation $b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3 + b_4\mathbf{v}_4 = \mathbf{0}$ has only the trivial solution (all $b_i = 0$). This equation can also be written as a homogeneous linear system with three equations and four unknowns (the b_i's). The coefficient matrix is $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 3 & 8 & 0 \\ 1 & 4 & 16 & 0 \end{bmatrix}$. This reduces to row echelon form, for instance $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 8 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, from which we can see there is a free variable in the solution. That means there is a nontrivial solution (there are infinitely many, but we only need one). One solution is $b_1 = 16, b_2 = -8, b_3 = 1, b_4 = 0$. So we have the nontrivial linear dependence relation

$$16\mathbf{v}_1 - 8\mathbf{v}_2 + 1\mathbf{v}_3 + 0\mathbf{v}_4 = \mathbf{0}, \tag{0.1}$$

   which proves that X is linearly dependent.

   (b) Now we want to express one of these vectors as a linear combination of the others. You do that by solving for a vector in Equation (0.1). For instance,

$$\mathbf{v}_1 = \frac{1}{2}\mathbf{v}_2 - \frac{1}{16}\mathbf{v}_3 + 0\mathbf{v}_4.$$

   (You could solve for $\mathbf{v}_2$ or $\mathbf{v}_3$ instead of $\mathbf{v}_1$ if you prefer, because in the linear dependence relation (0.1), the coefficient of each of these is not zero. You can't solve for $\mathbf{v}_4$ because that would mean dividing by 0.)

2. Find a smallest subset of X whose span equals $\mathrm{Span}(X)$.

   Answer: Since $\mathbf{v}_1$ is a linear combination of the other vectors in X, we don't need it to get the same span. In other words, $\mathrm{Span}\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \mathrm{Span}\, X$.

   Is $\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ the smallest we can get, or can we omit another vector? We can't omit any more. The reason is that $\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent. To prove that, set up the equation for a linear dependence relation: $b_2\mathbf{v}_2 + b_3\mathbf{v}_3 + b_4\mathbf{v}_4 = \mathbf{0}$. This is a homogeneous linear system with three equations and three unknowns. The coefficient matrix has rank 3, the number of columns. Therefore, the homogeneous system has a unique solution, which is the trivial solution (because a homogeneous system *always* has the trivial solution).

Quiz 11, 3/13.

$$\text{Let } X = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\mathbf{v}_4\}.$$

a) Is X a basis for $\mathbb{R}^3$?

Answer: (First method) Row reduction of the matrix $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix}$ gives a pivot position in three columns but not all.

(Second method) The matrix A has 3 rows and 4 columns. The number of pivot positions is at most 3 (the smaller of 3 and 4), so it's impossible for every column of A to be a pivot column.

(Both methods) In conclusion: Since not every column is a pivot column, X is not a linearly independent set, so it can't be a basis of anything.

b) Is X a basis for $\operatorname{Span} X$?

Answer: See part a).

c) Is $\operatorname{Span} X = \mathbb{R}^3$?

Answer: (First method) Yes, because every row of A has a pivot position.

(Second method) Yes, because X spans $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ (you can show that by explicitly writing each $\mathbf{e}_i$ as a linear combination of vectors in X; for instance $\mathbf{e}_1 = \mathbf{v}_1 - \mathbf{v}_4$) and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ spans $\mathbb{R}^3$ (it's the standard basis).

Quiz 12, 3/16.

$$A = \begin{bmatrix} 1 & 3 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 & 4 & 4 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$ What is a basis for each of the following?

(1) Col A
(2) Image(F_A)
(3) Nul A
(4) Kernel(F_A)
(5) Row A

Answer: (1) Since A is already in row echelon form you can see the pivot columns. The basis is those columns of A as matrices (i.e., vectors):

$$\text{Basis for Col } A = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \right\}.$$

Note that this is not Col$A$, which is the whole vector space, not just a basis.

Answer: (2) Image(F_A) = Col A so the answer is the same as to (2).

Answer: (3) The method is to solve the equation of $\operatorname{Nul} A$, which is $A\mathbf{x} = \mathbf{0}$ where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$. First, reduce A to reduced row echelon form; this is $A' = \begin{bmatrix} 1 & 3 & 0 & -1 & 0 & -4 \\ 0 & 0 & 1 & 1 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$.
The equation $A'\mathbf{x} = \mathbf{0}$ has the same solution set as $A\mathbf{x} = \mathbf{0}$ (they are equivalent linear systems; this is Chapter 1 stuff). The solution is

$$\begin{aligned} x_1 &= -3x_2 + 1x_4 + 4x_6, \\ x_2 &= 1x_2, \\ x_3 &= \qquad\quad - 1x_4 - 8x_6, \\ x_4 &= \qquad\quad\ \ 1x_4, \\ x_5 &= \qquad\qquad\qquad 1x_6, \\ x_6 &= \qquad\qquad\qquad 1x_6. \end{aligned}$$

(Since the pivot columns are columns 1, 3, 5, the free variables are x_2, x_4, x_6.) Write this as a matrix equation:

$$\mathbf{x} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 4 \\ 0 \\ -8 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

(This equation means that $\mathbf{x}$ is in the null space if and only if it equals a linear combination of this kind.) The basis is the set of these vectors:

$$\text{Basis for (3)} = \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -8 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(Theory says these vectors are linearly independent. You can see that by looking at rows 2, 4, and 6, which are the rows of the free variables.)

Answer: (4) $\operatorname{Kernel}(F_A) = \operatorname{Nul} A$ so the answer is the same as to (3).

Answer: (5) Remember that $\operatorname{Row} A = \operatorname{Col} A^T$. A basis for $\operatorname{Row} A$ is the set of nonzero columns of $(A')^T$. (You have to omit the zero columns if there are any.) The basis is

$$\text{Basis for (5)} = \left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ -1 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

Notice that you only have to do one row reduction (getting A into reduced row echelon form) to solve all five problems.

Quiz 13, 3/20.

Let $X = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}3\\2\\0\end{bmatrix}, \begin{bmatrix}4\\4\\0\end{bmatrix} \right\} = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$.

(1) Find a vector $\mathbf{u}$ in X such that $\mathrm{Span}(X - \{\mathbf{u}\}) = \mathrm{Span}(X)$.
(2) Find a basis B for $\mathrm{Span}(X)$ that is a subset of X.
(3) Find a vector $\mathbf{w} \in \mathbb{R}^3$ such that $B \cup \{\mathbf{w}\}$ is linearly independent.

Answer: The key is to set up the matrix

$$A = \begin{bmatrix}\mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{x}_4\end{bmatrix} = \begin{bmatrix}1 & 2 & 3 & 4\\0 & 1 & 2 & 4\\0 & 0 & 0 & 0\end{bmatrix}.$$

Then $\mathrm{Span}(X) = \mathrm{Col}(A)$. We want to put A into row echelon form but it already is, so that's easy.

For (1), pick a vector that's in a non-pivot column, such as $\mathbf{x}_4$. Then the column space is the same when you remove $\mathbf{x}_4$, so $\mathbf{u} = \mathbf{x}_4$ is an answer to (1).

For (2), you can choose the vectors in the pivot columns, so a basis is $B - \{\mathbf{x}_1, \mathbf{x}_2\}$. (There are other bases contained in X, but this is the easiest to find.)

For (3) we want any vector $\mathbf{w}$ that is not in $\mathrm{Span}(X)$. That is, $A\mathbf{x} = \mathbf{w}$ has no solution. Set up

$$\begin{bmatrix}A \mid \mathbf{w}\end{bmatrix} = \begin{bmatrix}1 & 2 & 3 & 4 & w_1\\0 & 1 & 2 & 4 & w_2\\0 & 0 & 0 & 0 & w_3\end{bmatrix}$$

and see what conditions are required to put $\mathbf{w} \in \mathrm{Span}(X)$. The condition is in row 3: we need $w_3 = 0$. So if $w_3 \neq 0$, $\mathbf{w}$ is not in $\mathrm{Span}(x)$ and $X \cup \{\mathbf{w}\}$ is linearly independent. For instance, let $\mathbf{w} = (0, 0, 2)$. (You might have noticed right away that a vector spanned by X must have $w_3 = 0$, but I'm using the general method to illustrate how it works.)

Quiz 14, 3/6.

List as many ways as you can think of to describe a subspace. For example, one way is as the column space of a matrix, $\mathrm{Col}\, A$. If you happen to duplicate an answer there is no penalty, except that it won't be counted twice.

2 points per correct answer. Can you give at least three answers?

Answer: Some answers are:

(a) As the column space of a matrix.
(b) As the row space of a matrix.
(c) As the null space of a matrix.
(d) By equations it satisfies.
(e) By a homogeneous linear system of which it is the solution set.
(f) As the image of a linear transformation.
(g) As the kernel of a linear transformation.

(h) By a basis.
(i) By a spanning set.

Quiz 15, 4/26.

A is an $n \times n$ matrix.

(1) Define an eigenvalue of A.
Answer: An eigenvalue is a scalar λ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero vector $\mathbf{x} \in \mathbb{R}^n$.
(2) Define an eigenvector of A.
Answer: An eigenvector is any nonzero vector $\mathbf{x}$ as in the answer to (1).
(3) Define the eigenspace of an eigenvalue λ of A.
Answer: The eigenspace of λ is the set $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \lambda\mathbf{x}\}$. It can also be defined as the set of all eigenvectors of λ together with the zero vector, $\mathbf{0}$ (you can't omit $\mathbf{0}$).

Quiz 16, 4/26.

Find the eigenvalues and their eigenvectors for $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.
Answer: Set up the characteristic equation: $\det(A - \lambda I) = 0$. Evaluate the determinant:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1v \end{vmatrix} \\ &= (1-\lambda)^3 + 1 + 1 - (1-\lambda) - (1-\lambda) - (1-\lambda) \\ &= (1 - 3\lambda + 3\lambda^2 - \lambda^3) + 2 - 3 + 3\lambda \\ &= \lambda^3 - 3\lambda^2 = \lambda^2(\lambda - 3). \end{aligned}$$

That gives the characteristic polynomial, whose zeros (or roots) are 0, 0, 3 (note that 0 has multiplicity 2; that's why I wrote it twice). These are the eigenvalues.

Eigenvectors of 0: Solve $(A - 0I)\mathbf{x} = \mathbf{0}$, i.e., $A\mathbf{x} = \mathbf{0}$. The solution is $\mathbf{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ where x_2, x_3 are any scalars (except $x_2 = x_3 = 0$ since $\mathbf{x} = \mathbf{0}$ is not technically an eigenvector). What is usually most useful is a basis for the eigenspace, which is $\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$.

Eigenvectors of 3: Solve $(A-3I)\mathbf{x}=\mathbf{0}$. The solution is $\mathbf{x}=x_3\begin{bmatrix}1\\1\\1\end{bmatrix}$ where x_3 is any scalar (except $x_3=0$). A basis for the eigenspace is $\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)$.

Quiz 17, 5/.

Quiz 18, 5/.