Quiz 3 for Math 304-08 2/13/2023 SOLUTIONS

Fully explain why your answer is correct, in every question.

(1) (10 points) The solution set to this linear system is a subset of $\mathbb{R}^3$. Is it a subspace of $\mathbb{R}^3$?

$$2x_1 + x_3 = 0$$
$$x_2 + x_3 = 4$$

Answer: No.
Reasons (any one is a good answer):

1. $\mathbf{0}$ is not in the solution space.

2. If $\mathbf{x} = (x_1, x_2, x_3)$ is in the solution space, and if t is a scalar, then $t\mathbf{x}$ should be in the solution space. We'll test this. $\mathbf{x}$ satisfies the linear system (because it is a solution, duh!) We test $t\mathbf{x} = (tx_1, tx_2, tx_3)$ for being a solution:

$$2(tx_1) + (tx_3) = 0?$$
$$(tx_2) + (tx_3) = 4?$$

Simplify:

$$t(2x_1 + x_3) = 0?$$
$$t(x_2 + x_3) = 4?$$

Using the fact that $\mathbf{x}$ is a solution:

$t(0) = 0$? Yes

$t(4) = 4$? Not for every scalar t.

Conclusion: Our set is not a subspace.

3. If $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ are in the solution space, then $\mathbf{x}+\mathbf{y}$ should be in the solution space. We'll test this. We test $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$ for being a solution:

$$2(x_1 + y_1) + (x_3 + y_3) = 0?$$
$$(x_2 + y_2) + (x_3 + y_3) = 4?$$

Rearrange:

$$(2x_1 + x_3) + (2y_1 + y_3) = 0?$$
$$(x_2 + x_3) + (y_2 + y_3) = 4?$$

Using the fact that $\mathbf{x}$ and $\mathbf{y}$ are solutions:

$(0) + (0) = 0$?

$(4) + (4) = 4$? No. So $\mathbf{x} + \mathbf{y}$ is not a solution.

Conclusion: Our set is not a subspace.

4. The system is inhomogeneous, therefore the solution set is not a subspace.

(2) (10 points) The solution set to this linear system is a subset of $\mathbb{R}^3$. Is it a subspace of $\mathbb{R}^3$?

$$\begin{aligned} 2x_1 + x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

Answer: Yes.
The reason (all three parts are necessary):

1. $\mathbf{0} = (\mathbf{0}, \mathbf{0}, \mathbf{0})$ is in the solution space because substituting $(x_1, x_2, x_3) = (0, 0, 0)$, we get equality. Conclusion: Our set may be a subspace.

2. If $\mathbf{x} = (x_1, x_2, x_3)$ is in the solution space, and if t is a scalar, then $t\mathbf{x}$ should be in the solution space. We'll test this. $\mathbf{x}$ satisfies the linear system (because it is a solution, duh!) We test $t\mathbf{x} = (tx_1, tx_2, tx_3)$ for being a solution:

$$\begin{aligned} 2(tx_1) + (tx_3) &= 0? \\ (tx_2) + (tx_3) &= 0? \end{aligned}$$

Simplify:

$$\begin{aligned} t(2x_1 + x_3) &= 0? \\ t(x_2 + x_3) &= 0? \end{aligned}$$

Using the fact that $\mathbf{x}$ is a solution:

$$\begin{aligned} &t(0) = 0? \text{ Yes} \\ &t(0) = 0? \text{ Yes, for every scalar } t. \end{aligned}$$

Conclusion: Our set may be a subspace.

3. If $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ are in the solution space, then $\mathbf{x}+\mathbf{y}$ should be in the solution space. We'll test this. We test $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$ for being a solution:

$$\begin{aligned} 2(x_1 + y_1) + (x_3 + y_3) &= 0? \\ (x_2 + y_2) + (x_3 + y_3) &= 0? \end{aligned}$$

Rearrange:

$$\begin{aligned} (2x_1 + x_3) + (2y_1 + y_3) &= 0? \\ (x_2 + x_3) + (y_2 + y_3) &= 0? \end{aligned}$$

Using the fact that $\mathbf{x}$ and $\mathbf{y}$ are solutions:

$$\begin{aligned} &(0) + (0) = 0? \\ &(0) + (0) = 0? \text{ Yes. So } \mathbf{x} + \mathbf{y} \text{ is a solution.} \end{aligned}$$

Conclusion: Our set is a subspace. It satisfies all three properties that a sub*set* of a vector space ($\mathbb{R}^3$) needs to satisfy to be a sub*space*.

Alternative reason:
The system is homogeneous, therefore the solution set is a subspace.

(3) (10 points) Invert this matrix: $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

Since we end up with the identity matrix I on the left, the matrix on the right is the inverse matrix:

$$A^{-1} = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$