QUIZ 5 FOR MATH 304-08, 3/1/2023 YOUR NAME ______________________________

Show all necessary reasoning and work for full credit.

(1) (6 points) Find the coordinate vector, $[p(t)]_{\mathcal{B}}$, of $p(t) = 5t^0 + 2t - 7t^2$ with respect to the standard basis $\mathcal{B} = \{t^0, t_1, t^2\}$ of $\mathcal{P}_2(t)$.

Answer: $[p(t)]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 2 \\ -7 \end{bmatrix}$, which are the coefficients of the basis vectors in $p(t)$.

(2) (4+6 points) A linear transformation $T : \mathcal{P}_2(t) \to \mathbb{R}^3$ is given by the rule

$$T(a_0t^0 + a_1t + a_2t^2) = \begin{bmatrix} a_0 - a_2 \\ 2a_2 \\ a_1 + a_0 \end{bmatrix}.$$

(a) Evaluate $T(3t - 5t^2)$.
Answer: In this polynomial, $a_0 = 0$, $a_1 = 3$, and $a_2 = -5$. Therefore,

$$T(3t - 5t^2) = \begin{bmatrix} 0 - (-5) \\ -5 \\ 3 + 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \\ 3 \end{bmatrix}.$$

(b) Find the coordinate vector, $[T(3t - 5t^2)]_{\mathcal{E}}$, [OMIT: of $p(t) = 5t^0 + 2t - 7t^2$] with respect to the standard basis $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbb{R}^3$.

Answer: First find $T(3t - 5t^2)$ (in part (a)). Then $[T(3t - 5t^2)]_{\mathcal{E}} = \begin{bmatrix} 5 \\ -5 \\ 3 \end{bmatrix}$,

because $T(3t - 5t^2) = 5\mathbf{e}_1 - 5\mathbf{e}_2 + 3\mathbf{e}_3$.

(3) (8+8+4 points) For the same linear transformation $T(a_0t^0 + a_1t + a_2t^2) = \begin{bmatrix} a_0 - a_2 \\ 2a_2 \\ a_1 + a_0 \end{bmatrix}$ as in the previous problem:

(a) Is T one-to-one (injective)?
Answer: Yes. You need to justify this. Injective means that if $T(p(t)) = T(q(t))$, then $p(t) = q(t)$. (See Ch. 10.) Let's write $p(t) = a_0t^0 + a_1t + a_2t^2$ and $q(t) = b_0t^0 + b_1t + b_2t^2$. Then

$$T(p(t)) = T(a_0t^0 + a_1t + a_2t^2) = \begin{bmatrix} a_0 - a_2 \\ 2a_2 \\ a_1 + a_0 \end{bmatrix}$$

and

$$T(q(t)) = T(b_0t^0 + b_1t + b_2t^2) = \begin{bmatrix} b_0 - b_2 \\ 2b_2 \\ b_1 + b_0 \end{bmatrix}.$$

We set these equal and see what results from that equation.

$$\begin{bmatrix} a_0 - a_2 \\ 2a_2 \\ a_1 + a_0 \end{bmatrix} = \begin{bmatrix} b_0 - b_2 \\ 2b_2 \\ b_1 + b_0 \end{bmatrix}$$

gives the linear system

$$\begin{aligned} a_0 - a_2 &= b_0 - b_2, \\ 2a_2 &= 2b_2, \\ a_1 + a_0 &= b_1 + b_0, \end{aligned}$$

from which it is easy to conclude that $a_2 = b_2$, therefore $a_0 = b_0$, and therefore $a_1 = b_1$. That is, $p(t) = q(t)$. Since $T(p(t)) = T(q(t))$ implies $p(t) = q(t)$, it follows that $p(t) \neq q(t)$ implies $T(p(t)) \neq T(q(t))$. Thus, T is injective.

(b) What is the range of T?

Answer: Remember that the range is the set of all $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ (the target) such that $T(a_0 t^0 + a_1 t + a_2 t^2) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ has a solution $a_0 t^0 + a_1 t + a_2 t^2 \in \mathcal{P}_2(t)$ (the domain). Using the formula for T, that means we want to find all $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that $\begin{bmatrix} a_0 - a_2 \\ 2a_2 \\ a_1 + a_0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, where the x's are given numbers, has a solution for the a's. (Be careful! The unknowns are the a's, *not* the x's!) We set up the augmented matrix and get a row echelon form:

$$\begin{bmatrix} 1 & 0 & -1 & x_1 \\ 0 & 0 & 2 & x_2 \\ 1 & 1 & 0 & x_3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & x_1 \\ 0 & 0 & 2 & x_2 \\ 0 & 1 & 1 & x_3 - x_1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & x_1 \\ 0 & 0 & 1 & \frac{1}{2}x_2 \\ 0 & 1 & 1 & x_3 - x_1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 0 & x_1 + \frac{1}{2}x_2 \\ 0 & 0 & 1 & \frac{1}{2}x_2 \\ 0 & 1 & 0 & x_3 - x_1 - \frac{1}{2}x_2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & x_1 + \frac{1}{2}x_2 \\ 0 & 1 & 0 & x_3 - x_1 - \frac{1}{2}x_2 \\ 0 & 0 & 1 & \frac{1}{2}x_2 \end{bmatrix}.$$

The only thing that matters here is that the echelon form has no pivot position in the last column (the "constants" column). Therefore the original equation is always solvable, no matter what vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ we started with. (We don't need an actual solution.) That means the range of T is all of $\mathbb{R}^3$.

(c) Is T surjective?

Answer: Yes, because its range is the same as its target.