(1) (15 points) A is an $m \times n$ matrix whose nullity is 55 and whose rank is 20.

(a) (5 points) Is it possible to find the value of n? If yes, what is n?

Answer: Yes, $n = 75$ because rank + nullity $= n$.

(b) (5 points) Is it possible to find the value of m? If yes, what is m?

Answer: No, because we only know $m \geq$ rank $= 20$.

(c) (5 points) If I reveal that the linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T_A(\mathbf{x}) = A\mathbf{x}$ is surjective (onto), can you find the value of m? If yes, what is m?

Answer: Yes, $m = 20$, the rank.
By surjectivity, $\text{Range}(T_A) = \mathbb{R}^m$.
For a matrix transformation, $\text{Range}(T_A) = \text{col}(A)$.
The dimension of $\text{col}(A)$ is always $\text{rank}(A) = 20$.
Thus, $\dim \text{Range}(T) = \dim \text{col}(A) = 20$.

(2) (28 points) For the linear transformation $T : \mathcal{P}_2(x) \to \mathbb{R}^4$ defined by $T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \\ p(2) \\ p(3) \end{bmatrix}$:

(a) (5 points) What is $T(2 - x - x^2)$?

Answer: $\begin{bmatrix} 2-0-0 \\ 2-1-1 \\ 2-2-4 \\ 2-3-9 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -4 \\ -10 \end{bmatrix}$.

(b) (10 points) Find the kernel ("null space") of T.

Answer: We want all polynomials $p(x) \in \mathcal{P}_2(x)$ such that $T(p(x)) = \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Substituting the formula for T, that means

$$\begin{bmatrix} p(0) \\ p(1) \\ p(2) \\ p(3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

In order to make sense of this we need to write out $p(x) = a_0 + a_1 x + a_2 x^2$. Then the equation becomes

$$\begin{bmatrix} a_0 \\ a_0 + a_1 + a_1 \\ a_0 + 2a_1 + 4a_2 \\ a_0 + 3a_1 + 9a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

That is a homogeneous linear system whose coefficient matrix is (with its reduction to reduced row echelon form)

$$\begin{bmatrix}1&0&0\\1&1&1\\1&2&4\\1&3&9\end{bmatrix} \to \begin{bmatrix}1&0&0\\0&1&1\\0&2&4\\0&3&9\end{bmatrix} \to \begin{bmatrix}1&0&0\\0&1&1\\0&1&2\\0&1&3\end{bmatrix} \to \begin{bmatrix}1&0&0\\0&1&1\\0&0&1\\0&0&2\end{bmatrix} \to \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix},$$

which gives the solution $a_0 = 0$, $a_1 = 0$, $a_2 = 0$. That is, $\ker(T) = \{\mathbf{0}\}$.

(c) (3 points) Is T injective (one-to-one)?

Answer: Since there is only one vector in the kernel, T is injective.

(d) (10 points) Are the values of T on the standard basis of the domain—that is, the values $T(1)$, $T(x)$, $T(x^2)$—linearly dependent or independent?

Answer: First, calculate the values:

$$T(1) = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad T(x) = \begin{bmatrix}0\\1\\2\\3\end{bmatrix}, \quad T(x^2) = \begin{bmatrix}0\\1\\4\\9\end{bmatrix}.$$

Now we decide whether they are linearly independent. One way to do that is to see what linear combinations give the $\mathbf{0}$ vector. We solve $c_1T(1)+c_2T(x)+c_3T(x^2) = \mathbf{0}$, that is,

$$c_1\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}0\\1\\2\\3\end{bmatrix} + c_3\begin{bmatrix}0\\1\\4\\9\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}.$$

This is a homogeneous linear system whose coefficient matrix is (coincidentally?) the same as that in part (b), which means the solution is $c_1 = c_2 = c_3 = 0$. Since only that trivial linear combination can give the zero vector, the three vectors are linearly independent.

Another way to prove linear independence is to show that the matrix of the three vectors, i.e., $\begin{bmatrix}1&0&0\\1&1&1\\1&2&4\\1&3&9\end{bmatrix}$, has a pivot position in every column. To decide that, you do the same row reduction. (I chose the first method to emphasize the relevance of linear combinations.)