Show full justification of every answer.

(1) (5 points) Calculate the reciprocal $1/(2-i)$ of the complex number $2-i$.
Solution.
$$\frac{1}{2-i} = \frac{2+i}{|2-i|^2} = \frac{1}{5}(2+i) = \frac{2}{5} + \frac{1}{5}i.$$
Or, let $\frac{1}{2-i} = a+bi$ so $(a+bi)(2-i) = 1+0i$, which simplifies to $(2a+b)+(2b-a)i = 1+0i$. That means $2a+b=1$ and $2b-a=0$, whose solution is $a=\frac{2}{5}$, $b=\frac{1}{5}$, so $\frac{1}{2-i} = \frac{2}{5}+\frac{1}{5}i$.

(2) (20 points) Diagonalize the matrix $\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$.
Solution. Step 1. Find the eigenvalues.
$$\begin{vmatrix} 3-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 + 1 = \lambda^2 - 6\lambda + 10.$$
Thus, $\lambda = \dfrac{6 \pm \sqrt{36 - 4\cdot 10}}{2} = 3 \pm i$.

Step 2. Find the eigenspaces, or rather, a basis for each eigenspace.

For $\lambda = 3+i$, the equation of the eigenspace is $(A-(3+i)I)\mathbf{x} = \mathbf{0}$, so we look for the solution by row reduction:
$$\begin{bmatrix} 3-(3+i) & 1 \\ -1 & 3-(3+i) \end{bmatrix} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \to \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \to \begin{bmatrix} 1 & i \\ 0 & 1+i^2 \end{bmatrix} = \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}.$$
Thus, the eigenspace is the solution space of $\begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}\mathbf{x} = \mathbf{0}$, whose equations are $1x_1 + ix_2 = 0$ and $0=0$ (I'm being excessively careful). That is, $x_1 = -ix_2$ so $\mathbf{x} = \begin{bmatrix} -ix_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} -i \\ 1 \end{bmatrix}$. The eigenspace is $\{x_2\begin{bmatrix} -i \\ 1 \end{bmatrix} : x_2 \in \mathbb{C}\}$ and a basis is $\{\begin{bmatrix} -i \\ 1 \end{bmatrix}\}$. The simplest choice of eigenvector is $\begin{bmatrix} -i \\ 1 \end{bmatrix}$.

For $\lambda = 3-i$, the equation of the eigenspace is $(A-(3-i)I)\mathbf{x} = \mathbf{0}$, so we look for the solution:
$$\begin{bmatrix} 3-(3-i) & 1 \\ -1 & 3-(3-i) \end{bmatrix} = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \to \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -i \\ 0 & 1+i^2 \end{bmatrix} = \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}.$$
Thus, the eigenspace is the solution space of $x_1 - ix_2 = 0$. That is, $x_1 = ix_2$ so $\mathbf{x} = \begin{bmatrix} ix_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} i \\ 1 \end{bmatrix}$. The eigenspace is $\{x_2\begin{bmatrix} i \\ 1 \end{bmatrix} : x_2 \in \mathbb{C}\}$ and the simplest choice of eigenvector is $\begin{bmatrix} i \\ 1 \end{bmatrix}$. Notice that this eigenvalue is the complex conjugate of the first one and its eigenvector is the complex conjugate of the first eigenvector (theorem).

Step 3. Find the diagonal matrix D and the matrix M (or P, same matrix). This is routine:
$$D = \begin{bmatrix} 3+i & 0 \\ 0 & 3-i \end{bmatrix}, \; M = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}, \text{ and } M^{-1} = \frac{1}{2}\begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix}.$$
Then $D = M^{-1}AM$, or equivalently $A = MDM^{-1}$.