Show full justification of every answer. There is only one question, but it has 4 parts.

(1) (5+5+15+5 points) Here is a set of vectors:

$$S = \{\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\} \subset \mathbb{R}^3.$$

In this problem use the dot product in $\mathbb{R}^3$.

(a) Is S an orthogonal set?
Solution. No, because not all its inner products are 0. E.g., $\mathbf{u}_1 \cdot \mathbf{u}_2 = 1$.

(b) Is S a basis for $\mathbb{R}^3$?
Solution. Yes. There are a few ways to prove this.

(c) Use the Gram–Schmidt process to turn S into an orthogonal set S'.

Solution. Step 1. $\mathbf{v}_1 = \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Step 2. $\mathbf{v}_2 = \mathbf{u}_2 - \dfrac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \dfrac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix}$.

Step 3. $\mathbf{v}_3 = \mathbf{u}_3 - \dfrac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \dfrac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \dfrac{2}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \dfrac{1}{3/2}\begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 1/3 \\ -1/3 \\ 1/3 \end{bmatrix}$.

Thus, $S' = \{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/3 \\ -1/3 \\ 1/3 \end{bmatrix}\}$.

(d) Is S' a basis for $\mathbb{R}^3$? You should be able to give a short reason.
Solution. Yes. Here are two short reasons.
(1) Because we began with a basis S (as shown in (b)).
(2) Because S' is an orthogonal set and has 3 vectors, which is the number needed for a basis of $\mathbb{R}^3$.