Math 323-05 Fall, 2015

Quiz Solutions

Quiz 1. (Sept. 1)

1. What is the equation of the xy-plane in 3-space, $\mathbb{R}^3$?

   Solution. $z = 0$.

   $(x, y, 0)$ is not a solution because the question asks for an equation. (An equation has an $=$ sign in it.)

2. What is the equation of a sphere in 3-space with center $(1, 2, p)$ and radius 9?

   Solution. $(x-1)^2 + (y-2)^2 + (z-p)^2 = 9^2$. Note the minus signs!

   If you think this is mysterious, think "Distance formula".

Quiz 2. (Sept. 2)

1. Prove that for any vector $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$.

   Solution. We compare the formula for $|\mathbf{u}|$ with that for the dot product. (If you don't compare formulas, you don't have a proof.)

   (1) $|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$ (definition of magnitude; this is the distance formula again).

   (2) $\mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2 + u_3^2$ (definition of dot product).

   Then combine these equations to get $|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$.

   Mistakes.

   Some people write $\mathbf{u}^2$. $\mathbf{u}$ isn't a number, so what do you mean by this?

2. Find the unit vector in the direction of $\mathbf{w} = \langle 1, 2, -2 \rangle$.

   Solution. The unit vector is $\frac{1}{|\mathbf{w}|}\mathbf{w} = \frac{1}{3}\langle 1, 2, -2 \rangle = \langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \rangle$.

   If you have any doubt, you can check the length of this vector: $|\langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \rangle| = \sqrt{(\frac{1}{3})^2 + (\frac{2}{3})^2 + (-\frac{2}{3})^2} = 1$. Just right!

   Mistakes.

   The "unit vector" is a vector. Finding $|\mathbf{w}|$ is a step in the right direction but it is not a vector.

   A picture doesn't tell me exactly what the vector is. You have to give the components in order to make it a specific vector.

   What's wrong with this? $\langle \mathbf{i}, 2\mathbf{j}, -2\mathbf{k} \rangle$.

   What's wrong with this? $\mathbf{w} = \sqrt{1^2 + 2^2 + (-2)^2} = 3$.

Quiz 3. (Sept. 8)

1. Find the area of the parallelogram determined by the vectors $\mathbf{a} = \langle 1, 3 \rangle$ and $\mathbf{b} = \langle 1, -3 \rangle$.

   Solution. We extend the vectors into 3-space by putting in a $z = 0$ coordinate. Then we take the cross product. So, let $\mathbf{a}' = \langle 1, 3, 0 \rangle$ and $\mathbf{b}' = \langle 1, -3, 0 \rangle$. Then

$$\mathbf{a}' \times \mathbf{b}' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 0 \\ 1 & -3 & 0 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ -3 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 1 & -3 \end{vmatrix} \mathbf{k} = 0\mathbf{i} - 0\mathbf{j} + (-6)\mathbf{k}.$$

   The volume is the magnitude of this vector, which is

$$|\mathbf{a}' \times \mathbf{b}'| = \sqrt{0^2 + (-0)^2 + (-6)^2} = 6.$$

(That is, the answer is 6.)

There are several other methods that work for this example (because it's in the xy-plane) but you need to know the method of 3-dimensional vectors. For one reason, plane geometry won't help if your vectors aren't in a coordinate plane.

2. Find the volume of the parallelepiped generated by the vectors

$$\mathbf{u} = \langle 1, 3, 0\rangle,\ \mathbf{v} = \langle 1, -3, 0\rangle, \text{ and } \mathbf{w} = \langle -1, -1, -1\rangle.$$

Solution. Use the triple product:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 3 & 0 \\ 1 & -3 & 0 \\ -1 & -1 & -1 \end{vmatrix} = (1)\begin{vmatrix} -3 & 0 \\ -1 & -1 \end{vmatrix} - (3)\begin{vmatrix} 1 & 0 \\ -1 & -1 \end{vmatrix} + (0)\begin{vmatrix} 1 & -3 \\ -1 & -1 \end{vmatrix}$$
$$= (1)(3) - (3)(-1) + (0)(-4) = 6.$$

The answer is $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = 6$.

There are many ways to write this solution, for instance you can write the vectors in any order, and you can have the cross product on the left side, for instance $(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u}$. You can also compute the cross product and then take the dot product.

In each method, you'll get either $+6$ or -6 as the value of the determinant (or the dot product), so you must take the absolute value.

Quiz 4. (Sept. 10)

1. What is the difference among these?

$$\langle 2, -3\rangle, \qquad (2, -3), \qquad \langle 2\mathbf{i}, -3\mathbf{j}\rangle, \qquad 2\mathbf{i} - 3\mathbf{j}, \qquad \langle 2\mathbf{i} - 3\mathbf{j}\rangle$$

Solution. The second in the list is a point (which has a definite location, but no direction or length), not a vector (which has no definite location, but has direction and length). The first and fourth are vectors. The third and fifth are meaningless because we haven't defined those notations.

The first and fourth are different ways to write the same vector. The first gives the components of the vector, which are scalars (numbers). (The mistake in the third is to give "components" that aren't numbers.) The fourth is a "linear combination" of the vectors $\mathbf{i}$ and $\mathbf{j}$; specifically, it's the same as $2\langle 1, 0\rangle - 3\langle 0, 1\rangle$, because of the definitions of $\mathbf{i}$ and $\mathbf{j}$. Now, let's apply vector algebra:

$$2\langle 1, 0\rangle - 3\langle 0, 1\rangle = \langle 2, 0\rangle + \langle 0, -3\rangle = \langle 2, -3\rangle.$$

The fifth is just a mistaken combination of the two different notations we use for a vector.

2. Given points $P = (1, 2, 3)$ and $Q = (1, 1, 1)$:
   (a) Find a parametric equation of the line PQ.

   Solution. We can specify a line with one point in it and a vector in the direction of the line. We need a direction vector; this can be $\overrightarrow{PQ} = \langle 1-1, 1-2, 1-3\rangle = \langle 0, -1, -2\rangle$. ($\overrightarrow{QP}$ is equally good. So is $2\overrightarrow{PQ}$, for another example.) For the point I choose P. (Q would be just as good.) Then the equation is

$$\langle x, y, z\rangle = \langle 1, 2, 3\rangle + t\langle 0, -1, -2\rangle,$$

where $t \in \mathbb{R}$ (that is, t varies over all real numbers).

You can get a different equation by choosing a different direction vector or base point, but your equation will always generate the same set of points. For instance, if you used $\overrightarrow{QP}$ instead of $\overrightarrow{PQ}$, your parametric equation would be

$$\langle x, y, z\rangle = \langle 1, 2, 3\rangle + t\langle 0, 1, 2\rangle,$$

where $t \in \mathbb{R}$. Some people did this.

Note that I've given what the book calls a "vector equation", but it's really just the vector form of parametric equation. The book's "parametric equations" are the same thing written as separate equations for the x, y, and z components.

(b) Find the coordinates of the point R such that $\overrightarrow{PR} = 3\overrightarrow{PQ}$.

Solution. The point R is $P + 3\overrightarrow{PQ} = (1, 2, 3) + 3\langle 0, -1, -2\rangle = (1, 0, -1)$.

I'm being a bit tricky with notation here when I add a vector to a point. The idea is to put the tail of the vector $3\overrightarrow{PQ}$ at the point P and see where the head of the vector is; that's R.

A better way is with the *position vector* or *radius vector* of a point. The position vector of point $P(x, y, z)$ is $\overrightarrow{OP} = \langle x, y, z\rangle$, the vector from the origin to the point. This vector is very useful. For instance, $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR}$ (by vector addition), so

$$\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OP} + 3\overrightarrow{PQ} = \langle 1, 2, 3\rangle + 3\langle 0, -1, -2\rangle = \langle 1, 0, -1\rangle.$$

That tells you the point R is $(1, 0, -1)$. (Notice that R is not a vector! $\overrightarrow{OR}$ is the vector.)

Quiz 5. (Sept. 16)

1. (a) Find a vector that is normal to the plane with the equation $x - y + 4z = 2$.

Solution. Read it off from the coefficients in the equation: $\mathbf{n} = \langle 1, -1, 4\rangle = \mathbf{i} - \mathbf{j} + 4\mathbf{k}$. (Both forms are equally correct, since they are two representations of the same vector.)

(b) Find a unit vector that is normal to the plane.

Solution. Make $\mathbf{n}$ into a unit vector in the same direction:

$$\mathbf{n}/|\mathbf{n}| = \frac{1}{3\sqrt{2}}\mathbf{n} = \frac{1}{3\sqrt{2}}\langle 1, -1, 4\rangle,$$

using the length of $\mathbf{n}$, which is $|\mathbf{n}| = \sqrt{1^2 + (-1)^2 + 4^2} = \sqrt{18}$.

2. Find a vector that is normal to the plane that contains the points $P(1, 0, 0)$, $Q(1, 2, 3)$, and $R(2, 2, 2)$.

Solution. If we can find two vectors in the plane (not parallel), we take their cross product. Since P, Q, R are not collinear, two vectors between them will do, e.g., $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ (or $\overrightarrow{PQ}$ and $\overrightarrow{QR}$, etc.). The vectors are $\overrightarrow{PQ} = \langle 0, 2, 3\rangle$ and $\overrightarrow{PR} = \langle 1, 2, 2\rangle$. Then a vector that answers the question is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 3 \\ 1 & 2 & 2 \end{vmatrix} = (4-6)\mathbf{i} - (0-3)\mathbf{j} + (0-2)\mathbf{k} = -2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}.$$

Quiz 6. (Sept. 18)

1. Find the vector projection of $\mathbf{u} = 3\mathbf{i} - 4\mathbf{j}$ onto $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

   Solution. We want

$$\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{-1}{3}\mathbf{v} = -\frac{1}{3}\langle 1,1,1\rangle = \langle -\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}\rangle.$$

   Make sure you project onto $\mathbf{v}$, not $\mathbf{u}$. That means we want a vector in the direction of $\mathbf{v}$; that is, we want a scalar multiple of $\mathbf{v}$.

   I used the formula $|\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v}$ because I like it; you don't have to use it.

Quiz 7. (Sept. 25)

1. What is the domain of the function $\mathbf{r}(t) = \dfrac{t}{\sin t}\mathbf{i} + \ln(t^2-3)\mathbf{j} + 3(\tan t)\mathbf{k}$?

   Solution. We need values of t at which all three components of the vector ($t/\sin t$, $\ln(t^2-3)$, and $3\tan t$) are defined. That means $\sin t \neq 0$, $t^2 - 3 > 0$, and $\tan t \neq \infty$.

   - $\sin t = 0$ when $t = n\pi$ for an integer n (zero, positive, or negative). So the domain excludes all $n\pi$.
   - $t^2 > 3$ when $t > \sqrt{3}$ or $t < -\sqrt{3}$, that is, the domain is at most $t \in (-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$.
   - $\tan t$ is infinite for $t = \pi/2 + n\pi$ for any integer n, so the domain excludes all such numbers.

   Putting it all together, the domain excludes all integer multiples of $\pi/2$. The answer is

$$\{t \in (-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty) : t \neq n\pi/2\}.$$

   A mistake is to assume t has to be in some interval like $[0, 2\pi]$. Nothing is said about such a restriction, so you shouldn't assume it.

2. For the curve $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j} + 12t\mathbf{k}$:

   (a) What is the point of the curve at $t = 1$?

   Solution. Just substitute $t = 1$. Since $\mathbf{r}(1) = 1\mathbf{i} + 1\mathbf{j} + 12\mathbf{k}$, the point is $(1, 1, 12)$.

   (b) What is the unit tangent vector to the curve at $t = 1$?

   Solution. First differentiate: $\mathbf{r}'(t) = \langle 2t, 3t^2, 12\rangle$. Then evaluate at $t = 1$: $\mathbf{r}'(1) = \langle 2, 3, 12\rangle$. Then make a unit vector: since $|\mathbf{r}'(1)| = \sqrt{2^2 + 3^2 + 12^2} = \sqrt{157}$, the unit vector is

$$\frac{1}{\sqrt{157}}\langle 2, 3, 12\rangle.$$

   An alternative is to find the unit vector $\mathbf{r}'(t)/|\mathbf{r}'(t)|$ in general and then substitute $t = 1$.

Quiz 8. (Oct. 2) Given a curve $\mathbf{r}(t)$ in $\mathbb{R}^3$.

1. Write the definition of the curvature of $\mathbf{r}(t)$.

   Solution. The curvature is defined as $\kappa := |d\mathbf{T}(s)/ds|$, where $s =$ arc length function. This is the rate at which the curve's direction ($\mathbf{T}(t)$) changes as the curve progresses. (It is independent of which parametrization you use.)

2. Write a formula for the curvature other than the definition.

   Solution.

   One answer is $|\mathbf{r}'(t) \times \mathbf{r}''(t)|/|\mathbf{r}'(t)|^3$.

   Another answer is $\left|\dfrac{\mathbf{T}(t)}{dt}\Big/\dfrac{ds}{dt}\right| = \left|\dfrac{\mathbf{T}(t)}{dt}\right|\Big/\left|\dfrac{ds}{dt}\right|$ or equivalently (by the formula for ds/dt) $\left|\dfrac{\mathbf{T}(t)}{dt}\right|\Big/\left|\dfrac{\mathbf{r}'(t)}{dt}\right|$. (This is also an acceptable answer to part (1), because it's a simple transformation of the actual definition using the chain rule.)

3. Write the definition of the unit normal vector to the curve $\mathbf{r}(t)$.

   Solution. This is the *direction* in which the curve's direction changes, in other words the direction towards which it is turning at t. (The *amount* of change is κ.) It is therefore the unit vector

$$\mathbf{N} := \frac{d\mathbf{T}(s)/ds}{|d\mathbf{T}(s)/ds|} = \frac{d\mathbf{T}(s)/ds}{\kappa}$$

   (two ways to write the same thing).

   (To compute $\mathbf{N}$ it's usually easier to use the chain rule to make all derivatives with respect to t, as I did for κ in the second answer to part (2).)

Quiz 9. (Oct. 23) Here is a plane: $2x - y - 3z + 4 = 0$. Here is a point $P = (9, -9, 0)$.

(a) Find the line through P that is perpendicular to the plane.

Solution. The direction of the line is the normal vector of the plane. The normal vector is $\langle 2, -1, -3\rangle$ (or any nonzero scalar multiple). The vector equation of the line is therefore $\langle x, y, z\rangle = \langle 9, -9, 0\rangle + t\langle 2, -1, -3\rangle$. Another way to write the same answer is $x = 9+2t,\ y = -9-t,\ z = -3t$. Another way to specify the line is as a set of points: $\{(x, y, z) : x = 9 + 2t,\ y = -9 - t,\ z = -3t,\ t \in \mathbb{R}\}$. Another way to give equations of the line is $(x-9)/2 = -y+9 = -z/3$.

Some people gave an equation of a plane. Some people only gave a vector, such as $\langle 2, -1, -3\rangle$. None of that answers the question because it doesn't specify a line.

Some people forgot to write an equation, just giving $\langle 9, -9, 0\rangle + t\langle 2, -1, -3\rangle$ as the answer; this is pretty good but the best answer states the line by an equation or as a set.

(b) Find a line through P that is parallel to the plane.

Solution. Parallel to the plane means perpendicular to the normal vector. Find a vector $\mathbf{v}$ that's perpendicular to $\langle 2, -1, -3\rangle$, for instance $\langle 1, 2, 0\rangle$. (How to decide perpendicularity? You'd better know that!) Then you use that direction vector and the point P to get an equation of a line (not a plane!), just as in (a): $\langle x, y, z\rangle = \langle 9, -9, 0\rangle + t\langle 1, 2, 0\rangle$. Or, you can express your line in any of the other valid ways.

Quiz 10. (Oct. 30)

1. Simplify as far as possible: $\sqrt{4+9t^2+t^4}$.

   Solution. The only possible simplification is $\sqrt{(t^2+\frac{9}{2})^2-\frac{65}{4}}$. That is useful if you're integrating the square root since it has the form $\sqrt{u^2-a^2}$. Otherwise, there's no useful simplification of this expression.

2. (a) Evaluate as far as possible: $\begin{vmatrix} 2 & 0 & 4 \\ -1 & -3 & 4 \end{vmatrix}$.

   Solution. This is a meaningless expression that looks mathematical but isn't. Our book never defines the determinant of a non-square array. There might be a definition in some corner of higher math, but I don't know any.

   (b) Is your result a number, a vector, or neither?

   Solution. It's *nothing*—there is no result!

3. Evaluate the integrals:

   (a) $\displaystyle\int_0^{x^2}\int_x^3 xy^2\,dy\,dx.$

   Solution. $\int_0^{x^2}\int_x^3 xy^2\,dy\,dx = \int_0^{x^2} x\frac{y^3}{3}\Big|_{y=x}^3\,dx = \int_0^{x^2} x(27-\frac{x^3}{3})\,dx = 27\cdot\frac{x^3}{3}-\frac{x^5}{15}\Big|_{x=0}^{x^2} = 9x^6-\frac{x^{10}}{15}$. The x in the upper limit of the outer integral has nothing to do with the x of the integrand of the outer integral; this is just a potentially confusing shorthand that is sometimes convenient. Therefore, the result is a function of the x that is in the upper limit of the outer integral. The result is not a number.

   (b) $\displaystyle\int_0^{y^2}\int_x^3 xy^2\,dy\,dx.$

   Solution. $\int_0^{y^2}\int_x^3 xy^2\,dy\,dx = 27\cdot\frac{x^3}{3}-\frac{x^5}{15}\Big|_{x=0}^{y^2} = 9y^6-\frac{y^{10}}{15}$. The y in the upper limit of the outer integral has nothing to do with the y of the inner integral. The result is a function of the y in the upper limit of the outer integral.

   (c) What would you say if I told you they were supposed to give numerical values?

   Solution. You could say, "What trick are you trying to pull?" You might (cleverly) say there has to be something wrong with setting up those integrals, because as they're written, they can't give numbers.

Quiz 12. (?) $\mathbf{F}(x,y)$ is a vector field in $\mathbb{R}^2$.

1. Define the property of being conservative.

   Solution. $\mathbf{F}$ is the gradient of some scalar function $f(x,y)$.

   That's the only definition. However, there is another property which is partially correct; it is $\dfrac{\partial Q}{\partial x}=\dfrac{\partial P}{\partial y}$, where $\mathbf{F}=P\mathbf{i}+Q\mathbf{j}$; that is, P is the x-component and Q is the y-component of the vector field.

2. Is $\mathbf{F}(x,y)=x\mathbf{i}+xy\mathbf{j}$ conservative?

   Solution. It is not, because $\frac{\partial P}{\partial y}=\frac{\partial x}{\partial y}=0$ and $\frac{\partial Q}{\partial x}=\frac{\partial(xy)}{\partial x}=y$ are not equal.

Quiz 13. (?)

1. Which of the following expressions is integrated in Green's Theorem?
   (a) $\dfrac{\partial P}{\partial x} - \dfrac{\partial Q}{\partial y}$
   (b) $\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}$
   (c) $\dfrac{\partial Q}{\partial y} - \dfrac{\partial P}{\partial x}$
   (d) $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$

   Solution: (d). This is important! There is a trick for remembering it: you compute $\nabla \times \mathbf{F}$ and look at the coefficient of $\mathbf{k}$, which is exactly (d). We'll see more about this later.
2. In which direction does the curve go in Green's Theorem?
   a. Clockwise.
   b. Counterclockwise.

   Solution: (b), assuming the region is inside the curve. If the region is outside the curve, it's (a). You keep the region on the left if you're going in the "positive" direction on its boundary.

Quiz 14. (Nov. 20) Is this vector field conservative?

$$\mathbf{F}(x, y) = \left\langle \frac{y}{\sqrt{x^2 + y^2}}, \frac{-x}{\sqrt{x^2 + y^2}} \right\rangle$$

Solution. Let's compare partial derivatives. In this vector field, $P = \frac{y}{\sqrt{x^2+y^2}}$ and $Q = \frac{-x}{\sqrt{x^2+y^2}}$. The partial derivatives are

$$\frac{\partial P}{\partial y} = \frac{-x^2}{(x^2 + y^2)^{3/2}} \quad \text{and} \quad \frac{\partial Q}{\partial x} = \frac{y^2}{(x^2 + y^2)^{3/2}}.$$

They are not equal, so $\mathbf{F}$ cannot be conservative.

It's essential to compute the partial derivatives correctly. Many people got the wrong partials; then comparing them is useless.

Quiz 15. (Nov. 23)

1. A region D is $\{(x, y) \in \mathbb{R}^2 \mid 4 \le x^2 + y^2 \le 9,\ y > -1\}$.
   Give a precise reason for every answer, as far as you are able to.
   (a) Sketch D.
   (b) Is D open?
   (c) Is D closed?
   (d) Is D connected?
   (e) Is D simply connected?

Solution.

D is a partial ring, cut off at the bottom before it closes. It's important to make a pretty good picture because otherwise you'll find it hard to answer all the questions.

D is not open, because it contains boundary points, for instance $(0, 3)$, which satisfies $x^2 + y^2 = 9$, a weak inequality, and also has $y > -1$. There is no tiny circle around a boundary point of D that is entirely wtthin D.

On the other hand, it is not closed, because it does not contain all of its boundary; for instance, a point $(x, -1)$ with suitable value of x is a boundary point and is not in D because D satisfies $y > -1$ (strict inequality). By the way, we never defined closed and we don't need the definition, but "closed" does not mean "not open". (Sorry if that sounds strange. It's math. Technically, a set is closed if it contains all its boundary points. Our set D contains some but not all boundary points.)

You can see in a picture that D is connected. For any two points in it, there is a path in D that joins the points. *The path can curve at will. It does not have to be a straight line segment.*

You can also see D is simply connected, because any closed curve you draw in D cannot enclose any points that are not in D.

2. What does the following sentence mean to you? (This is a separate question; it has nothing to do with D.)

   "The set is not open, connected, and simply connected."

   Solution. This sentence is ambiguous. There is no right answer. Is the set not open, while it is connected and simply connected? Or is it not open, not connected, and not simply connected? I got this as an answer on several papers and I could not decide what those people meant.

   I didn't give a grade for this question. It was intended to show you that such a sentence could be confusing.

Quiz 15a. (Nov. 23, not collected)

Is the vector field $\mathbf{F}(x, y) = \dfrac{-y}{\sqrt{x^2 + y^2}}\mathbf{i} + \dfrac{x}{\sqrt{x^2 + y^2}}\mathbf{j}$ conservative?

Solution. The natural first step is to compare partial derivatives. Letting $P = \dfrac{-y}{\sqrt{x^2 + y^2}}$ and $Q = \dfrac{x}{\sqrt{x^2 + y^2}}$ as usual, we get

$$\frac{\partial P}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial Q}{\partial x}.$$

Does that mean $\mathbf{F}$ is conservative? *No!* It could go either way. There's another condition that, if satisfied does guarantee $\mathbf{F}$ is conservative: the vector field must be defined on a domain that is simply connected.

If the domain is simply connected, the partial derivative condition does prove $\mathbf{F}$ is conservative. But this vector field is not defined at $(0, 0)$ so its domain is $\mathbb{R}^2 - \{(0, 0)\}$ (that's short notation for $\{(x, y) \in \mathbb{R}^2 \mid (x, y) \neq (0, 0)\}$), which is not simply connected, e.g., a circle around the origin encloses a point that is not in the domain.

If the domain is not simply connected, we still can't easily tell whether $\mathbf{F}$ is conservative, but there is a way that can prove it is not conservative. Find a closed curve C on which the integral $\oint_C \mathbf{F} \cdot d\mathbf{r} \neq 0$. If so, then $\mathbf{F}$ is not conservative. (If that integral is 0, we still don't know the answer about $\mathbf{F}$.)

In our example, take a circle of radius a around the origin and compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$. The answer is π, which is not 0. So, $\mathbf{F}$ is not conservative *on the domain* $\mathbb{R}^2 - \{(0,0)\}$.

There is a way to get around this problem. If you reduce the domain so it is simply connected, $\mathbf{F}$ may become conservative on that domain. For instance, let's define our vector field only for $y > 0$. That subset of the plane is simply connected (and open and connected) so $\mathbf{F}$ is conservative *on that set*, by one of the theorems in Chapter 16. I know it seems weird that whether a vector field is conservative or not can depend on how you choose the domain. It is weird! It also leads to some famous mathematics.

Quiz 16. (Dec. 1) (Problem 16.6 # 33 from the last homework assignment.) Find an equation of the tangent plane to the parametric surface $x = u + v$, $y = 3u^2$, $z = u - v$ at the point $(3, 2, 0)$.

Solution. We need a normal vector so we compute $\mathbf{r}_u$ and $\mathbf{r}_v$. Since

$$\mathbf{r} = \langle u + v, 3u^2, u - v \rangle,$$

we get

$$\frac{\partial \mathbf{r}}{\partial u} = \langle 1, 6u, 1 \rangle, \qquad \frac{\partial \mathbf{r}}{\partial v} = \langle 1, 0, -1 \rangle.$$

Then a normal vector is

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -6u, 2, -6u \rangle.$$

Now (but this could be done sooner: any time after you calculated $\mathbf{r}_u$ and $\mathbf{r}_v$) we have to evaluate this at the point $(3, 2, 0)$. To do that, we have to find u and v at that point. It's easy by setting $3 = x = u + v$ and $0 = z = u - v$, whose solution is $u = \frac{3}{2}$, $v = \frac{3}{2}$. Therefore, at $(3, 2, 0)$,

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -9, 2, -9 \rangle.$$

The equation of the tangent plane is $-9(x - 3) + 2(y - 2) - 9z = 0$, or in another form,

$$9x - 2y + 9z = 23.$$