Math 323 TEST 1 9/30/2015 SOLUTIONS

- Hand in *both this paper and the test booklets.*
- SHOW ALL NECESSARY WORK FOR YOUR SOLUTION. To get full credit, you must show all the work you did and it must be enough to justify your answer.
- Begin each new solution on a fresh page.
- Simplify all answers as far as possible.
- Make sure you mark your answer clearly. If I can't distinguish your intended answer, I won't give credit for it.
- No calculators or other electronic aids!

1. [Points: 1/2] Did you read and will you follow the instructions? Answer ____________

   Solution: The particular instruction I looked for was to begin each numbered solution on a fresh page. It's also important to simplify answers.

2. [Points: 19] A parallelogram $CABD$ (two of whose sides are AB and AC) has vertices $A(0,1,2)$, $B(2,3,4)$, $C(1,3,5)$, and D.

   (a) [Points: 3] What are the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$?

   Solution: $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \langle 2,2,2\rangle$ and $\overrightarrow{AC} = \langle 1,2,3\rangle$.

   (b) [Points: 3] Express $\overrightarrow{CD}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

   Solution: $\overrightarrow{CD} = \overrightarrow{AB}$ because they are parallel vectors of equal length (by properties of a parallelogram). Note that I'm not asking for numerical components and I'm not asking for $\overrightarrow{CD}$ in terms of any other vectors, like $\overrightarrow{BD}$ or $\overrightarrow{OC}$, etc.

   (c) [Points: 3] Use (b) to find the coordinates of D.

   Solution: Add $\overrightarrow{CD}$ to C, or in vector language, $\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD} = \overrightarrow{OC} + \overrightarrow{AB} = \langle 1,3,5\rangle + \langle 2,2,2\rangle = \langle 3,5,7\rangle$, so $D = (3,5,7)$.

   (d) [Points: 10] Find the area of the parallelogram.

   Solution: Area $= |\overrightarrow{AB} \times \overrightarrow{AC}|$.

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} = \mathbf{i}(6-4) - \mathbf{j}(6-2) + \mathbf{k}(4-2) = \langle 2,-4,2\rangle.$$

   So the area $= \sqrt{2^2 + (-4)^2 + 2^2} = \sqrt{24} = 2\sqrt{6}$.

   Fairly common errors: Dot product. Wrong vectors in the cross product. Taking half the value (that's for a triangle). Silly arithmetic errors due to haste. *Don't rush!*

3. [Points: 20] $\mathbf{u} = \langle 2,-1,-2\rangle$ and $\mathbf{v} = \langle 4,-5,0\rangle$.

   (a) [Points: 13] Find a (nonparametric) equation of the plane through the point $A(1,1,1)$ that is parallel to both lines $\mathbf{r} = \mathbf{u} + t\mathbf{v}$ and $\mathbf{r} = \mathbf{v} + t\mathbf{u}$.

   Solution: The lines have directions $\mathbf{v}$ and $\mathbf{u}$, respectively. Therefore, the plane has to contain those directions. This implies that $\mathbf{u} \times \mathbf{v}$ is a normal vector to the plane. Thus,

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -2 \\ 4 & -5 & 0 \end{vmatrix} = \mathbf{i}(0-10) - \mathbf{j}(0+8) + \mathbf{k}(-10+4) = \langle -10, -8, -6 \rangle.$$

The equation of the plane is then $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \langle 1, 1, 1 \rangle$, which is $-10x - 8y - 6z = -24$; of course you can negate and you can divide by 2, giving equally valid answers. Note that $\mathbf{r} := \langle x, y, z \rangle$, as usual.

(b) [Points: 7] Find a parametric equation of the same plane.

Solution: $\mathbf{r} = \langle 1, 1, 1 \rangle + s\mathbf{u} + t\mathbf{v} = \langle 1 + 2s + 4t, 1 - s - 5t, 1 - 2s \rangle$.

I didn't expect anyone to get this; the expected grade was 0. It was discussed in class but it doesn't show up in the book until Chapter 14 (I think). If you did get it, congratulations.

4. [Points: 15] Here are two points in $\mathbb{R}^3$: $A(0, 0, 0)$ and $B(0, 0, 6)$.

(a) [Points: 10] Find an equation for the set S of all points $P(x, y, z)$ such that the distance of P from A is 2 times the distance of P from B.

Solution: First, the basic equation stated in the problem is $|\overrightarrow{AP}| = 2\,|\overrightarrow{BP}|$. Then you translate this into coordinates:

$$\sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = 2\sqrt{(x-0)^2 + (y-0)^2 + (z-6)^2}.$$

You have to simplify this; the first step is to square both sides: $x^2 + y^2 + z^2 = 4(x^2 + y^2 + (z-6)^2)$, which becomes

$$x^2 + y^2 + z^2 = 4x^2 + 4y^2 + 4(z^2 - 12z + 36).$$

Now simplify and collect terms: $3x^2 + 3y^2 + 3z^2 - 48z + 144 = 0$, which you can divide by 3 to get

$$x^2 + y^2 + z^2 - 16z + 48 = 0.$$

You should now recognize this as the equation of a sphere, because all of the squared variables have the same coefficient (i.e., 1). If you want to verify that, you complete the square involving z, thus,

$$0 = x^2 + y^2 + z^2 - 16z + 48 = x^2 + y^2 + z^2 - 16z + 8^2 - 16,$$

which simplifies to the standard equation of a sphere,

$$x^2 + y^2 + (z-8)^2 = 16.$$

That shows the sphere has center $(0, 0, 8)$ and radius 4.

I gave credit for $|\overrightarrow{AP}| = 2 \cdot |\overrightarrow{BP}|$ but in the future I expect you to *simplify as far as possible* (see directions!).

Some people left out the length symbols on the vectors. The equation $\overrightarrow{AP} = 2\,\overrightarrow{BP}$ is (a) false and (b) useless for getting an answer. Some people think $\sqrt{x^2 + y^2 + z^2} = x + y + z$ or something similar; get rid of that idea as fast as you can.

(b) [Points: 5] What kind of surface is S?

Solution: A sphere. (Specifically, it has center $(0, 0, 8)$ and radius 4.)

5. [Points: 35] A curve is given by the parametric equation $\mathbf{r}(t) = \langle a\cos t, a\sin t, t\rangle$, where $t \in \mathbb{R}$. (a is a positive constant.)

(a) [Points: 5] Prove that the curve lies in the cylindrical surface $x^2 + y^2 = a^2$.

Solution: You just prove that the values $x(t), y(t), z(t)$ satisfy the surface's equation. Thus:

$$x(t)^2 + y(t)^2 = a^2\cos^2 t + a^2\sin^2 t = a^2(\cos^2 t + \sin^2 t) = a^2,$$

which proves the curve does satisfy that equation.

(b) [Points: 10] Find the unit tangent vector $\mathbf{T}$ at every point on the curve.

Solution: The easy tangent vector is

$$\mathbf{r}'(t) = \frac{d}{dt}\langle a\cos t, a\sin t, t\rangle = \langle -a\sin t, a\cos t, 1\rangle.$$

The unit tangent vector is $\mathbf{r}'(t)/|\mathbf{r}'(t)|$. The length $|\mathbf{r}'|$ is

$$\sqrt{(-a\sin t)^2 + (a\cos t)^2 + 1^2} = \sqrt{a^2(\sin^2 t + \cos^2 t) + 1} = \sqrt{a^2+1}.$$

Thus,

$$\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|} = \frac{1}{\sqrt{a^2+1}}\langle -a\sin t, a\cos t, 1\rangle.$$

Anyone who thinks $\sqrt{a^2+1} = a+1$ is having a serious algebra problem!

(c) [Points: 10] Find the tangent line at $t = 3\pi/2$.

Solution: I hope you noticed the word *line*. If not, you wasted many points.

A tangent vector (it need not be a unit vector) is

$$\mathbf{r}'(3\pi/2) = \langle -a\sin(3\pi/2), a\cos(3\pi/2), 1\rangle = \langle -a(-1), a(0), 1\rangle = \langle a, 0, 1\rangle.$$

This is the direction of the line. (You must simplify the trig functions to get full credit. Read the directions!) Now we need to find the point at which this is tangent, i.e., where $t = 3\pi/2$. That point is

$$\mathbf{r}(3\pi/2) = \langle a\cos(3\pi/2), a\sin(3\pi/2), 3\pi/2\rangle = \langle a(0), a(-1), 3\pi/2\rangle = \langle 0, -a, 3\pi/2\rangle.$$

The parametric equation of the line is then

$$\mathbf{r} = \langle 0, -a, \frac{3\pi}{2}\rangle + s\langle a, 0, 1\rangle = \langle sa, -a, \frac{3\pi}{2} + s\rangle,$$

where s is the parameter of the line (I could have used t, but it's a different t from the one used in the curve so I chose a different letter). Either form of equation of the line is fine. (Here $\mathbf{r} = \langle x, y, z\rangle$, denoting the position vector of any point on the line.)

You could give symmetric equations of the line. I happen to like the parametric form and I think it's easier.

(d) [Points: 10] Find a point on the curve where the tangent line is parallel to the xz-plane.

Solution: We need the y-component of $\mathbf{r}'(t)$ (the tangent vector from (b)) to be zero, so the tangent vector is parallel to the xz-plane. Where is that the case? You could just pick the point from (c), i.e., $\langle 0, -a, \frac{3\pi}{2}\rangle$, since the tangent vector there is $\langle a, 0, 1\rangle$. Or you could find all such points: since we need $y'(t) = 0$, that means $a \cos t = 0$, so $\cos t = 0$. The complete solution is $t = \pi/2 + n\pi$, $n \in \mathbb{Z}$.

You only need to give one solution, for instance, $t = \pi/2$; but do not forget to answer the question, which asks for the *point*, not the value of t. So, if we like $t = \pi/2$, the point is $\mathbf{r}(\pi/2) = \langle 0, a, \pi/2 \rangle$.

6. [Points: 10] Let $\mathbf{f}(t)$ and $\mathbf{g}(t)$ be vector functions of t. Use your knowledge of vector algebra and calculus to find the derivative of $\mathbf{f}(t) \times (\mathbf{f}(t) + \mathbf{g}(t))$ in simplest form, without using components in your calculations. (Partial credit if your solution is longer than necessary. Partial credit if you use components.)

Solution 1: You can simplify the expression using vector algebra.

$$\mathbf{f}(t) \times (\mathbf{f}(t) + \mathbf{g}(t)) = \mathbf{f}(t) \times \mathbf{f}(t) + \mathbf{f}(t) \times \mathbf{g}(t) = \mathbf{0} + \mathbf{f}(t) \times \mathbf{g}(t)$$

(since any vector cross-multiplied by itself equals $\mathbf{0}$) $= \mathbf{f}(t) \times \mathbf{g}(t)$. Then differentiate using the product rule:

$$\frac{d}{dt}(\mathbf{f}(t) \times \mathbf{g}(t)) = \mathbf{f}'(t) \times \mathbf{g}(t) + \mathbf{f}(t) \times \mathbf{g}'(t).$$

This is *not* the same as $\mathbf{f}'(t) \times \mathbf{g}(t) + \mathbf{g}'(t) \times \mathbf{f}(t)$, which is wrong. Cross product is not commutative.

Solution 2: You can differentiate first:

$$\begin{aligned}\frac{d}{dt}[\mathbf{f}(t) \times (\mathbf{f}(t) + \mathbf{g}(t))] &= \mathbf{f}'(t) \times (\mathbf{f}(t) + \mathbf{g}(t)) + \mathbf{f}(t) \times (\mathbf{f}'(t) + \mathbf{g}'(t)) \\ &= \mathbf{f}'(t) \times \mathbf{f}(t) + \mathbf{f}'(t) \times \mathbf{g}(t) + \mathbf{f}(t) \times \mathbf{f}'(t) + \mathbf{f}(t) \times \mathbf{g}'(t) \\ &= \mathbf{f}'(t) \times \mathbf{f}(t) + \mathbf{f}(t) \times \mathbf{f}'(t) + \mathbf{f}'(t) \times \mathbf{g}(t) + \mathbf{f}(t) \times \mathbf{g}'(t) \\ &= \mathbf{f}'(t) \times \mathbf{f}(t) - \mathbf{f}'(t) \times \mathbf{f}(t) + \mathbf{f}'(t) \times \mathbf{g}(t) + \mathbf{f}(t) \times \mathbf{g}'(t)\end{aligned}$$

(cross product is anticommutative)

$$\begin{aligned}&= \mathbf{0} + \mathbf{f}'(t) \times \mathbf{g}(t) + \mathbf{f}(t) \times \mathbf{g}'(t) \\ &= \mathbf{f}'(t) \times \mathbf{g}(t) + \mathbf{f}(t) \times \mathbf{g}'(t).\end{aligned}$$

I gave a lot of detail here; you don't have to write all that. I think either method is easier to do correctly and gives a better answer than using the components of the vectors—provided you know your vector algebra.

7. [Points: 1/2] Did you read and follow the instructions? Answer ____________