Math 323 TEST 2 10/28/2015 SOLUTIONS

1. [Points: 1/2] You can write your answer on the test paper.

2. [Points: 10+10+5] A surface has equation $f(x, y, z) = 2$, where $f(x, y, z) = x^3+3y^2-9z^2$.
   Grading note: In (a, b) I gave 6 points for $\mathbf{r}'(t)$ (which is needed for both parts), 7 points for the tangent plane, and 7 points for the directional derivative.

   (a) Find an equation of the tangent plane to the surface at the point $(2, 1, -1)$.

   Solution: The first step is to find the gradient:
$$\nabla f(x, y, z) = \langle 3x^2, 6y, -18z\rangle.$$
   Then evaluate it at the point $(2, 1, -1)$: $\nabla(2, 1, -1) = \langle 12, 6, 18\rangle$. Then use this vector as the coefficients in the equation of a plane that contains the point $(2, 1, -1)$:
$$\langle 12, 6, 18\rangle \cdot \langle x, y, z\rangle = \langle 12, 6, 18\rangle \cdot \langle 2, 1, -1\rangle,$$
   simplifying to
$$12x + 6y + 18z = 12,$$
   which you may further simplify to $2x + y + 3z = 2$.

   Notice that the value $2 = f(x, y, z)$ that determines this level surface has no role in the solution.

   Notice that I do not use the general form $\nabla f(x, y, z)$ in writing the tangent plane equation. That can cause errors because the x, y, z in the gradient are completely different from the x, y, z in the tangent plane equation. I put in the numbers first, that is, $\nabla f(2, 1, -1)$. If you want to see how this looks in general, the point where we get the tangent plane should be called (x_0, y_0, z_0) and the gradient at that point is $\nabla f(x_0, y_0, z_0)$; then the equation of the tangent plane is $\nabla f(x_0, y_0, z_0) \cdot \langle x, y, z\rangle = \nabla f(x_0, y_0, z_0) \cdot \langle x_0, y_0, z_0\rangle$. All the subscripted quantities are supposed to be known values, while x, y, z are variables subject to this equation.

   (b) Find the directional derivative of the function $f(x, y, z)$ at $(2, 1, -1)$ in the direction of $\langle 1, 1, -1\rangle$.

   Solution: We need two things: the gradient at $(2, 1, -1)$, which we already know, and the unit vector in the direciton of $\mathbf{u}$, which is $(1/\sqrt{3})\langle 1, 1, -1\rangle$. Thus, the directional derivative is
$$\nabla(2, 1, -1) \cdot \frac{1}{\sqrt{3}}\langle 1, 1, -1\rangle = \frac{4}{\sqrt{3}}.$$

   (c) Find all points on the surface where the tangent plane is horizontal.

   Solution: A horizontal tangent plane means the gradient is vertical, i.e., it has the form $\nabla f(x, y, z) = \langle 0, 0, k\rangle$, where $k \neq 0$. Comparing to the gradient equation, that means $3x^2 = 0$ and $6y = 0$, thus $x = y = 0$. So, we need a point $(0, 0, z)$ *on the surface*. It satisfies the equation $0^3 + 3 \cdot 0^2 - 9z^2 = 2$, in other words, $z^2 = -2/9$. But that's impossible. So the answer is: There are no such points!

3. [Points: 19] Let $f(x,y) = x^2 - xy + 2y^4$. Find all local minima and maxima of $f(x,y)$ in the xy-plane, both the points and their function values.

Solution: First step: The critical points are where the gradient $= \mathbf{0}$. The gradient is $\nabla f(x,y) = \langle 2x - y, -x + 8y^3 \rangle$. (Remember, we're in 2 dimensions.) Therefore, $y = 2x$ and $x = 8y^3$. You eliminate one of these variables, getting (for instance) $x = 64x^3$. Then *factor*: $64x^3 - x = 0$ (*Never divide by x!*), so $x(64x^2 - 1) = 0$, so $x = 0$ or $x = \pm 1/8$. Since $y = 2x$, the critical points are $(0,0)$, $(1/8, 1/4)$, and $(-1/8, -1/4)$.

Now test the critical points using the second-partials determinant.

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 24y^2 \end{vmatrix} = 48y^2 - 1.$$

(Be careful to do the second partials correctly. Partials are tricky when you're in a hurry.) Also, keep $f_{xx} = 2$ in mind. Now here are the results for the three critical points:

Point	D	f_{xx}	Type	Function Value
$(0,0)$	$-1 < 0$		Saddle point	not required
$(\frac{1}{8}, \frac{1}{4})$	$48(\frac{1}{4})^2 - 1 = 2 > 0$	$2 > 0$	Local min.	$-\frac{1}{128}$
$(-\frac{1}{8}, -\frac{1}{4})$	$48(-\frac{1}{4})^2 - 1 = 2 > 0$	$2 > 0$	Local min.	$-\frac{1}{128}$

Addendum: You can prove by algebra that this function has absolute minima (they are the local minima) and no absolute maximum, but that isn't required.

4. [Points: 10] Evaluate the integral $\iint_D x^2 \sin y \, dA$, where D is the region bounded by $x = 0$, $y = 3\pi$, and $x = 2y$.

Solution: You should draw the region; though it isn't required, it makes the setup safer. However, I don't have good computer drawing facilities so I'm omitting that. Description: It's a triangle with vertices at $(0,0)$, $(0,3\pi)$, and $(6\pi, 3\pi)$. The left side is $x = 0$, the top is $y = 3\pi$, and the bottom right is $x = 2y$, or $y = \frac{1}{2}x$ if you prefer.

Method 1: Put the y-integral inside: $\iint_D x^2 \sin y \, dA = \int_0^{6\pi} \int_{\frac{1}{2}x}^{3\pi} x^2 \sin y \, dydx$. We get the limits of integration from the outside in: in D, the lowest value of x is 0 and the highest is 6π. These limits cannot depend on y because y will disappear in computing the inner integral. Having set up the limits for x, we have to find the range of y for each value of x in the interval $[0, 6\pi]$. A value of x gives a vertical line. That line intersects D in an interval. The low end of the interval is at the lower boundary of D, i.e., $y = \frac{1}{2}x$. The high end of the interval is at the top of D, i.e., $y = 3\pi$. Those are the lower and upper limits of integration for y (in the inner integral).

Method 2: Put the x-integral inside: $\iint_D x^2 \sin y \, dA = \int_0^{3\pi} \int_0^{2y} x^2 \sin y \, dxdy$. From the outside in: in D, the lowest value of y is 0 and the highest is 3π. These limits cannot depend on x because x will disappear in computing the inner integral. Now we have to find the range of x for each value of y in the interval $[0, 3\pi]$. A value of y gives a horizontal line. That line intersects D in an interval. The left end of the interval is at $x = 0$. The right end of the interval is at $x = 2y$. Those are the lower and upper limits of integration for x (in the inner integral).

I will now do the inner integration in Method 1: $\int_0^{6\pi} \int_{\frac{1}{2}x}^{3\pi} x^2 \sin y \, dydx = \int_0^{6\pi} \left[x^2(-\cos y)\right]_{y=\frac{1}{2}x}^{3\pi} dx = \int_0^{6\pi} x^2[-\cos 3\pi + \cos \frac{1}{2}x] \, dx = \int_0^{6\pi} x^2[1 + \cos \frac{1}{2}x] \, dx$. The main point is to remember that, while you integrate with respect to y, x is like a constant. Don't get confused by its appearing to be a letter (of course it is a letter, but it doesn't vary while you integrate). (Method 2's inner integral is also simple.)

I gave 6 points for correctly setting up an iterated integral. Then 2 points for the inner integration. What remains is the outer integration; since that requires integration by parts (twice in both methods) and is not important for this course, I gave only 2 points for doing it, and I didn't expect it when setting the grading guidelines.

5. [Points: 15+10+10+10] A space curve is given by the formula $\mathbf{r}(t) = \langle t, \frac{3}{\sqrt{2}}t^2, 3t^3 \rangle$ for $t \geq 0$.

(a) Find the arc length function $s(t)$ measured from $t = 0$.

Solution: The arc length function is $s(t) = \int_0^t |\mathbf{r}'(t)| \, dt$. So we have to compute $|\mathbf{r}'(t)|$ first. $\mathbf{r}'(t) = \langle 1, 3\sqrt{2}\, t, 9t^2 \rangle$, so

$$|\mathbf{r}'(t)| = \sqrt{1^2 + (3\sqrt{2}\, t)^2 + (9t^2)^2} = \sqrt{1 + 18t^2 + 81t^4} = \sqrt{(1 + 9t^2)^2} = 1 + 9t^2.$$

It's essential to see that you can simplify the square root; otherwise integrating is long and complicated. Now the integral is easy:

$$s(t) = \int_0^t 1 + 9t^2 \, dt = t + 3t^3.$$

That's the answer to (a).

Note that the $|\mathbf{r}'(t)|$ in the integral has to be the function, not the value at $t = 0$ or 1, and I did not ask for the arc length from $t = 0$ to a specific number like $t = 1$. Also, arc length (being a length) is *not a vector.*

(b) Find the unit tangent vector $\mathbf{T}(t)$ at the point where $t = 1$.

Solution: Using the definition,

$$\mathbf{T}(t) = \frac{1}{|\mathbf{r}'|}\mathbf{r}' = \frac{1}{1 + 9t^2}\langle 1, 3\sqrt{2}\, t, 9t^2 \rangle.$$

Substituting $t = 1$, we get

$$\mathbf{T}(1) = \frac{1}{10}\langle 1, 3\sqrt{2}, 9 \rangle.$$

(c) Find the normal vector $\mathbf{N}$ at the same point.

Solution: The definition says that $\mathbf{N}$ is the unit vector in the direction of $\mathbf{T}'(t)$, i.e., $\mathbf{N}(t) = \mathbf{T}'(t)/|\mathbf{T}'(t)|$. You cannot differentiate $\mathbf{T}(1)$; you must use a general formula for $\mathbf{T}(t)$. Fortunately, I have that in (b). I use the quotient formula for the derivative, with the result that

$$\begin{aligned}\mathbf{T}'(t) &= \frac{1}{(1+9t^2)^2}\left[(1+9t^2)\langle 0, 3\sqrt{2}, 18t\rangle - 18t\langle 1, 3\sqrt{2}\,t, 9t^2\rangle\right] \\ &= \frac{1}{(1+9t^2)^2}\left[\langle 0, 3\sqrt{2}+27\sqrt{2}t^2, 18t+162t^3\rangle - \langle 18t, 54\sqrt{2}\,t^2, 162t^3\rangle\right] \\ &= \frac{1}{(1+9t^2)^2}\langle -18t, 3\sqrt{2}-27\sqrt{2}\,t^2, 18t\rangle.\end{aligned}$$

Therefore,

$$\mathbf{T}'(1) = \frac{1}{100}\langle 18, -24\sqrt{2}, 18\rangle = \frac{6}{100}\langle 3, -4\sqrt{2}, 3\rangle$$

and

$$|\mathbf{T}'(1)| = \frac{6}{100}\sqrt{3^2+4^2\cdot 2+3^2} = \frac{3\sqrt{2}}{10}.$$

Finally, we can write that

$$\mathbf{N}(1) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\sqrt{2}}{10}\langle 3, -4\sqrt{2}, 3\rangle.$$

I didn't expect people to do all this. I did expect A students, but not C students, to know that $\mathbf{N} = \mathbf{T}'/|\mathbf{T}'|$, but that was just for 2 points.

(d) What is the curvature of this curve at $t = 1$? Remember to simplify your answer as far as you can.

Solution:

Method 1: After doing all that calculation for (c) it's easy to use the formula $\kappa = |\mathbf{T}'|/|\mathbf{r}'| = \frac{3\sqrt{2}}{10}/10 = \frac{3\sqrt{2}}{100}$.

Method 2: If I had not done the calculations for (c), I could still use the formula $\kappa = |\mathbf{r}' \times \mathbf{r}''|/|\mathbf{r}'|^3$. It's pretty easy if you've done (a). First, the cross product:

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3\sqrt{2}\,t & 9t^2 \\ 0 & 3\sqrt{2} & 18t\end{vmatrix} = 27\sqrt{2}\,t^2\mathbf{i} - 18t\mathbf{j} + 3\sqrt{2}\,\mathbf{k} = 3(9\sqrt{2}\,t^2\mathbf{i} - 6t\mathbf{j} + \sqrt{2}\,\mathbf{k})$$

because $\mathbf{r}''(t) = \langle 0, 3\sqrt{2}, 18t\rangle$. Thus, at $t = 1$,

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{3\sqrt{162+36+2}}{10^3} = \frac{3\sqrt{200}}{1000} = \frac{3\sqrt{2}}{100}.$$

(This is the same answer as from Method 1, so I'm sure it's correct.)

I gave 2 points for knowing either of the formulas for κ. The other 8 points are for doing the calculations.

6. [Points: 1/2] You can write your answer on the test paper.