Math 330-05 Project Your name ________________________________

Rules and Regulations.

- The project must be entirely your own work. You may consult the textbook and your class notes. Do not consult any other book or article or online source. Do not work with anyone else and do not use a computer except for simple arithmetic.
- I don't expect you to solve every task completely. (Some can be, some can't be.) Just do your best. Submit partial solutions if you have them.
- I will give credit based on how good your work is. If you don't solve the task but your work is smart, you will get points. If you explain your solution well, you get points. If you don't explain your solution at all, you will not get points.
- Write legibly. (If I can't read your work, I can't give credit.)

Description.

In each task I will give you a recursive definition of a sequence $(x_1, x_2, \dots, x_n, \dots)$. I will ask you to answer some question(s) about that sequence for different starting values $x_1 = m$, $m \in \mathbb{N}$. The sequences are all similar to the $3x + 1$ and $x + 1$ sequences, but they are not those sequences.

A *cycle* in a sequence is a subsequence that repeats. (E.g., in the $3x + 1$ problem, there is the cycle $(1, 4, 2, 1, \dots)$ that repeats the same values. No one has found any other cycle in the $3x + 1$ problem, no matter what initial value m they tried.) In each task, there may be cycles for many initial values, or no cycles at all, or in between. Whether there is a cycle may depend on the initial value m.

Tasks.

Task 1. *The* $2x+1$ *sequence.* A sequence begins with a natural number m, that is, $x_1 = m$. Then for $n \geq 2$,

$$x_n = \begin{cases} 2x_{n-1} + 1 & \text{if } x_{n-1} \text{ is odd,} \\ \frac{1}{2}x_{n-1} & \text{if } x_{n-1} \text{ is even.} \end{cases}$$

The task is to find, for each initial value m, a formula for x_n in terms of n and m.

Hint: The *even part* of a natural number m is the highest power of 2 that divides m. The *odd part* of m is the largest odd number that divides m. The even part of any odd number m is $1 = 2^0$ (that's the highest power of 2 that divides an odd number) and the odd part is m itself. For an even number m, the even part is 2^e where $e > 0$ and the odd part is $m/2^e$. For example, the even part of 9 is $1 = 2^0$ and the odd part of 9 is 9. $36 = 2^2 \cdot 9$ so the even part of 36 is $4 = 2^2$ and the odd part of 36 is 9. The even part of $32 = 2^5$ is $32 = 2^5$ and the odd part is 1.

Task 2. *The* $3x-1$ *sequence.* (This is not the $3x+1$ sequence.) A sequence begins with a natural number m, that is, $x_1 = m$. Then for $n \geq 2$,

$$x_n = \begin{cases} 3x_{n-1} - 1 & \text{if } x_{n-1} \text{ is odd,} \\ \frac{1}{2}x_{n-1} & \text{if } x_{n-1} \text{ is even.} \end{cases}$$

The task is to find as many different cycles as you can. Each initial value m either ends with a cycle, or never ends.

Note: If you find the same numbers in a cycle, it will be the same cycle. For instance, if $x_7 = 10$ then you will have $x_8 = 5, x_9 = 14, x_{10} = 7, x_{11} = 20, x_{12} = 10, x_{13} = 5$, and the cycle will be $5, 14, 7, 20, 10, 5$ (and repeat). We say this is the same cycle as $10, 5, 14, 7, 20, 10$ and $7, 20, 10, 5, 14, 7$, etc.

It is possible that some initial values m do not lead to cycles at all. If you find such an initial value, include that in your solution; it will be interesting. If you can't decide about some initial value m that you try, after computing many terms of the sequence $(x_n)_1^\infty$, that will also be interesting; include it in your solution.

Computer searches are possible but you won't get more credit for (for example) 100 cycles than for 10 cycles unless you find a pattern.

Task 3. *The* $4x+1$ *sequence.* A sequence begins with a natural number m, that is, $x_1 = m$. Then for $n \geq 2$,

$$x_n = \begin{cases} 4x_{n-1} + 1 & \text{if } x_{n-1} \text{ is odd,} \\ \frac{1}{2}x_{n-1} & \text{if } x_{n-1} \text{ is even.} \end{cases}$$

The task is to find a formula for x_n in terms of n and m (similarly to Task 1) if you can, or to find sequences with cycles (as in Task 2) if you can, or to find out whatever you can about the sequences for different initial values m.