MATH 330-05 Quiz 3 Your name Solutions

Show all answers and work on this paper. Show all the work necessary to solve the problem. (That could be zero work.)

= **sign.** You should know what "=" means. For example, $n + 1 = n^2 + 2n + 1$ is a false statement. Also, "=" is not math for "is".

(1) (15 points) Write a complete proof *using mathematical induction* of the following proposition.

**Proposition Q3.** $n^2 > n$ for all natural numbers $n \geq 2$.

**Solution** [with explanations].

[Step 1. State the induction hypothesis. 3 points.]
Let $S(n)$ be the statement that $n^2 > n$.
[This is one statement for each separate $n$, not one statement for all $n$. To define $S(n)$ it does not matter what $n$ is. "Let $S(n)$ be the statement that $n^2 > n$ for $n \geq 2$." is not an induction hypothesis, it is the whole proposition, or at best it is ambiguous and sloppy writing.]

[Step 2. Settle the base case (or cases). 4 points.]
Base case: Since the proposition says $n \geq 2$, the base case should be $n = 2$, not $n = 1$. [I put $n \geq 2$ on purpose because you need to pay attention to the range of $n$ in the proposition. Note that $S(1)$ is false.]
$S(2)$ is $2^2 > 2$. This says $4 > 2$, which is true. So $S(2)$ is true.

Induction step [7 points]: Let $n \geq 2$. [First, state the range of $n$ for this step. 2 points.]
Assume $S(n)$ is true, that is, $n^2 > n$. [Essential, or you are not doing induction. 1 point.]
We want to prove $S(n+1)$, that is, $(n+1)^2 > n+1$. [It is recommended to state the objective. You have not proved it yet; you cannot use it to prove itself.]
To prove that, we calculate the difference:

$$(n + 1)^2 - (n + 1) = n^2 + 2n + 1 - n - 1 = n^2 + n > 2n$$

(because we assumed $n^2 > n$)

$$> 0$$

because $n \geq 2 > 0$. Therefore, $(n + 1)^2 > n + 1$. [3 points]
We have proved $S(n + 1)$. [1 point. Note: we have not really proved $S(n + 1)$. We have only proved $S(n + 1)$ assuming $S(n)$. But I accept the short form.]

[State the conclusion. 1 point.]
We conclude that $S(n)$ is true for all $n \geq 2$.

TURN OVER FOR QUESTION 2.

(2) (10 points) Define the relation $R$ on the set $\mathbb{Z}$ by $xRy$ if $x^2 = y^2$. It is a fact (don't prove it) that $R$ is an equivalence relation. Find the equivalence classes of $R$. Prove the correctness of your answer.

**Solution** [with explanations].

[The instructions say DO NOT PROVE that $R$ is an equivalence relation. If you proved it, you wasted your time.]

There is an equivalence class $[n]$ for each $n \in \mathbb{Z}$. By definition of an equivalence class,

$$[n] = \{x \in \mathbb{Z} : x^2 = n^2\}.$$

Since $x^2 = n^2$ has the solution $x = \pm n$, the equivalence class is

$$[n] = \{n, -n\}.$$

[This gives all equivalence classes because you considered all elements $n$ in $\mathbb{Z}$.]

[5 points. This is the first step in the solution. No credit for the general definition $[x] = \{y \in A : y \sim x\}$.]

[Now we have to find out which classes are redundant, since if $[n] = [m]$ then we should only list one of them. 5 points.]

We see that $[n] = [-n]$ so we only need one of them; so we can list only the equivalence classes of nonnegative numbers. The equivalence classes are

$$[0] = \{0, -0\}$$

which should be simplified to $\{0\}$, and

$$[n] = \{n, -n\} \text{ for each } n \in \mathbb{N}.$$

[For example, $[1] = \{1, -1\}$, $[2] = \{2, -2\}$, etc.]