Math 381
Solution to Problem C3
16 Feb 2011

0.1. **The problem.**

Let G be a graph which has a cycle. Let C be a cycle in G and let e be any edge in C. Let a, b be arbitrary vertices in G. Explain in detail why, if G has a path connecting a to b, so does $G - e$.

0.2. **The solution.**

The proof starts by taking a specific path P from a to b in G, and ends by producing a specific path Q from a to b in $G - e$. The idea is to modify P as necessary to get Q.

If $e \notin P$, then $Q = P$ works, as Q is a path in $G-e$. So, from now on assume that $e \in P$. (Notice that I don't care whether P intersects C, or where a and b are. The only thing that matters is whether the particular edge e is in P.)

The path P starts at a and goes on to the first vertex in e, say x. (Call that part of P, from a to x, P_1.) Then it crosses e to the other endpoint, y, and continues on to b. (Call P_2 the part of P from y to b.) $C - e$ is a path from x to y. We can take the following walk W: P_1 from a to x, then $C - e$ from x to y, then P_2 from y to b. That will be a walk from a to b that avoids e, so it is in $G - e$. The only problem is that it is not necessarily a path. (There are many ways W can intersect itself and therefore fail to be a path.)

If you got this far, you did pretty well. If you noticed this complication, you did very well. But how do we handle it?

There are two ways to handle this difficulty. (1) We reduce the walk W to a path by cleverly cutting out parts that repeat. Or, (2) We change the rule for making the walk without e, so it will be a path. I will do the second method.

Let u be the first vertex in P that belongs to C. (So u is in P_1. It might be a or x, or it might be a different vertex.) Let v be the last vertex in P that belongs to C. (So v is in P_2; it might be y or b or neither of them.) Let's make a new walk: Take the following walk W': P from a to v, then take the part of $C - e$ that goes from u to v, and then P from v to b. I claim this walk is a path. The part of it that is in P cannot have a repeated vertex, because P has none. The part that is in $C - e$ also cannot, because $C - e$ has none. The only possible repeated vertex is a vertex in P that is used in W' and is also in C. But the only vertices of that type are u and v. (u was the first vertex of P in C, and v was the last, and we don't use any of P from u to v.) So, there can't be a repeated vertex in W'.

So, W' is a path from a to b in $G - e$. We proved that if there is a path from a to b in G, then there is also a path from a to b in $G - e$. Q.E.D.

0.3. **Remark.**

The solution here is not an efficient proof. Instead, I wrote it to explain how someone might come up with the idea of such a proof. If you take out all the explanation, it will be more efficient but it won't be as helpful to you, I think.