Solution to Problem D1

Problem Statement.

Suppose the average degree of a connected graph G is > 2. [The original question should have said G is connected.]

(a) Prove that G contains at least 2 cycles.

(b) Prove that it is possible for G to have exactly 2 cycles.

Solution to (a).

The first key step is that the average degree $= \dfrac{2q}{p}$. The reason is that the average degree is the total degree divided by the number of vertices. The total degree is $2q$ (the first theorem in the book), thus the average degree is $2q/p$. That means what we know from the assumption can be expressed as: $2q/p > 2$, or

$$q > p.$$

Next step: Recognize that a connected graph that has no cycles, i.e., a tree, has $q = p - 1$. (Another main theorem.) Therefore, G is not a tree; as it is connected, it has a cycle. Let C_1 be a cycle in G. Now we remove one edge e_1 from C_1. That gives a graph $G_2 := G - e_1$, in which C_1 does not exist (we deleted one of its edges).

But (the third step) we know how many edges G_2 has. In fact, q_1 (the number of edges in G_2) is $q_2 = q - 1$. Therefore, $q_2 \geq p$. Also, G_2 is connected. (We had a theorem that deleting an edge from a cycle in a connected graph leaves a graph that is still connected. That's what Problem C3 means! Do you see how?) Consequently, G_2 is either a tree—but it has too many edges since $q_2 > p - 1$, so that's impossible—or it contains a cycle. Let C_2 be a cycle in G_2. Then G contains the cycles C_1 and C_2. That nearly solves the problem.

There's one little detail to make the solution complete. We should explain why C_1 and C_2 are not the same. The reason is that $e_1 \in C_1$ while $e_1 \notin C_1$ because $C_2 \subseteq G_2 = G - e_1$.

Solution to (b).

Here is an example: Two cycles, with one common vertex.

Another: Two disjoint cycles, with a path connecting a vertex in one of them to a vertex in the other one.

In both cases, all vertices have degree 2 except for one or two that have higher degree. Thus, the average degree is > 2. Obviously, each graph has exactly two cycles.

Problem D1(c). Prove there are no other (connected) examples. (I'm confident this is true but I didn't try to find a proof. That's for you.)