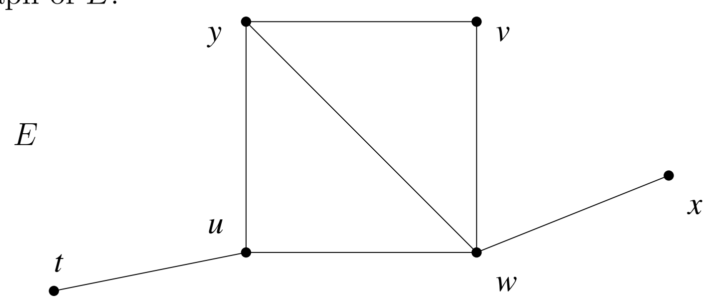
- Show all your work for each problem; show enough work to fully justify your answer.
- Work on the test paper when you can save time by not redrawing a diagram.
- Start each numbered problem on a *fresh page.*
- Hand in *both* this paper and test booklet.
- *Do not use the Four-Color Theorem* to solve any problems.

1. [Points: 1] Did you follow all the directions? ____________

Solution: Full credit if and only if you did follow the directions. I don't actually care whether your answer is correct!

2. [Points: 10] Find the line graph of E.



Solution: This is supposed to test whether you know what a line graph is. You do have to be careful.

Don't confuse line graph with dual graph. They are completely different!

3. [Points: 5] Find a planar graph G whose line graph $L(G)$ is nonplanar.

Solution: The simplest example is $G = K_{1,5}$, which is obviously planar, since $L(K_{1,5}) = K_5$, which I assume everyone knows is nonplanar. Other nice or not-so-nice examples are possible, such as W_4 (not so obvious) and W_n for $n \geq 5$ (whose line graph contains K_5 because there's a vertex of degree ≥ 5). The important things in this problem are that you know how to construct $L(G)$ and that you prove your example is correct (if it isn't as obvious as $G = K_{1,5}$).

4. [Points: 10]
(a) Find a largest graph with 8 vertices and diameter 5.
(b) How many edges are there?

Solution: This requires thinking about how to get diameter 5 (you start with P_5, which has 6 [not 5] vertices) and keep it as you add vertices (2 more, x and y). You can make an extra vertex x adjacent to at most 3 *consecutive* vertices of the P_5, because if x is adjacent to vertices at distance 3 or more along the path, then you shorten the distance between the endpoints. A good proof notices that. Then, if you choose the right 3 consecutive vertices for x and y you can add the edge xy for 12 edges. The careful construction proves this is the maximum. Your graph proves it is attainable.

(Some people didn't say anything about *consecutive* vertices. That was acceptable, though not quite as good, if you explained that having x adjacent to more than 3 vertices on P_5 shortens the diameter, if you deduced there could be no more than 12 edges, and if you had the example to prove 12 edges are possible.)

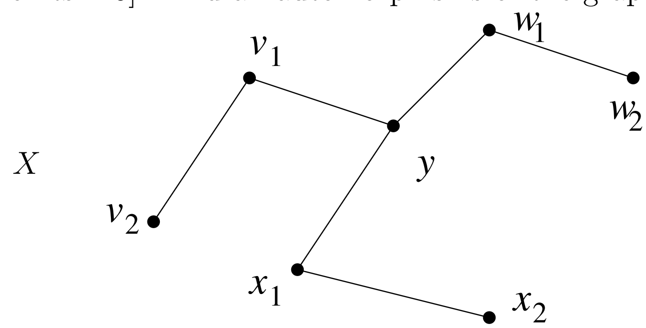
Example [4 pts.], proof [5 pts.], checking the number of edges [1 pt.]. Your graph must be reasonable to get credit. That means you had to have some idea of what you were doing and not just be guessing.

A minor deduction for missing the twelfth edge.

5. [Points: 10]
(a) What is the largest graph with $4n+2$ vertices and no K_5 subgraph?
(b) How many edges does it have?

Solution: This is a straightforward application of Turan's Theorem (you should get a 4-partite graph [5 pts.]), together with being able to figure out that the Turan graph for $p = 4n+2$ is $K_{n+1,n+1,n,n}$ [3 pts.] and that it has $6n^2+6n+1$ edges [2 pts.].

6. [Points: 10] Find all automorphisms of the graph X shown here.



Solution: An automorphism is a *function* $f : V \to V$, not a description. Thus, you should give the specific function for each automorphism, as a function like $f(y) = y$ etc., or as in $y \to y$ etc., or in a table of values of the various automorphisms. There are 6 automorphisms but just the number is not a complete answer; you have to state ("find") the automorphisms [5 pts.].

There should be a proof [5 pts.]. If you only gave an intuitive description of the automorphisms, you won't have a proof. A complete proof notes that an automorphism must preserve degrees, therefore $f(y) = y$. After that, there are various ways to go. In fact, you could start by examining the degree-1 vertices, or the degree-2 vertices, and get to y later.

7. [Points: 10] Prove that every planar graph can be colored in 6 colors. (This is vertex coloring.)

Solution: This was supposed to be a give-away problem! It appears that people who skipped this in the homework should have asked about it in class—but no one did.

The key is that any planar graph has a vertex of degree 5 or less. You didn't need to prove that; it's a separate theorem.

You use induction on the order of the planar graph. You find a vertex x in G with $d(x) \leq 5$, color $G \setminus x$ in 6 colors (justified by induction), and extend the coloring to x (possible due to $d(x)$).

Classic False Proof. If a planar graph G could not be colored in 4 (oops, I mean 6) colors it would have a subgraph isomorphic to K_5 (I mean K_7), but K_5 is nonplanar (therefore, so is K_7) so G cannot contain it as a subgraph. Therefore every planar graph is colorable in 4 (I mean 6) colors.

If this worked we wouldn't be still talking about the Four Color Theorem. Where is the flaw? (Every graph theorist gets this "proof" in the mail once in a while.)

Unreasonable proof. Don't use the 5 Color Theorem (5CT) to prove the 6CT, unless you prove the 5CT.

8. [Points: 10]
(a) Use Euler's Polyhedral Formula to prove that a planar graph satisfies $q \leq 3(p-2)$ if $p \geq 3$.
(b) Is this inequality satisfied by every planar graph with $p = 2$?

Solution: (a) [9 pts.] There are a lot of little steps worth 1 or 2 points each. It's not easy to remember all of them but some are important.

Know that Euler's formula says $p - q + r = 2$ for a plane *drawing* of any *connected* graph. It doesn't say that for a disconnected plane drawing. [2 pts. for Euler, stated correctly.]

If your proof is very good, you are showing where the assumption $p \geq 3$ is used (not easy!). [1 pt.]

The book has a proof starting with maximal planar graphs; you can look it up. Here is a different proof, which starts with connected planar graphs.

Step 1 [8 pts.]: Assume G is a connected planar graph with $p \geq 3$. Draw it (without crossings).

Each region A has at least 3 edges on its boundary (counting both sides if the edge has A on both sides), because otherwise G would have a loop or double edge (but G is a graph), or it would have one edge on its boundary twice and that would be the only edge (so $G = K_2$, contradicting $p \geq 3$).

Therefore,

sum of boundary lengths of all regions $\geq$ sum of 3 for all regions $= 3r$.

Also

sum of boundary lengths of all regions $= 2q$,

as each edge counts twice (once for each side). Therefore, $2q \geq 3r$, so $r \leq \frac{2}{3}q$. (You should explain how you get $2q \leq 3r$, or $2q = 3r$ for a maximal planar graph.)

Now we use this to eliminate r from Euler's formula:

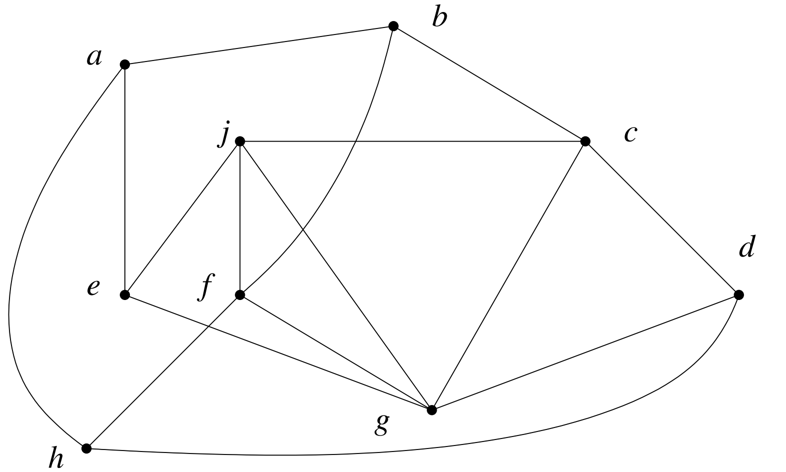
$$2 = p - q + r \leq p - q + \frac{2}{3}q = p - \frac{1}{3}q.$$

Solving for q, we get $q \leq 3(p-2)$.

Step 2 [1 pt.]: Assume G is a disconnected planar graph with $p \geq 3$. Add just enough edges to connect G, forming G'. This new graph is still planar. (You could add the new edges in a plane drawing of G.) Now G' is connected and planar with $p' = p \geq 3$. Thus, $q' \leq 3(p'-2)$, which implies $q \leq q' \leq 3(p-2)$.

Solution: (b) [1 pt.] No. K_2 fails.

9. [Points: 15]
(a) Is the graph H planar?
(b) Find its crossing number.



H

Solution: (a) [10 pts.] There are several subgraphs that are isomorphic to subdivisions of $K_{3,3}$. One (at least) is $K_{3,3}$ without subdivision. Finding any of these proves nonplanarity.

Solution: (b) [5 pts.] One can redraw the graph with 2 crossings. Thus, $\mathrm{cr}(H) \geq 1$ and $\mathrm{cr}(H) \leq 2$. You can't decide between these values because there is an edge, ae, such that $H \setminus ae$ is planar. Therefore, you can't use any method we know to prove the crossing number is not 1.

A good answer is "1 or 2". If you tried the edge-deletion method correctly, that's even better. However, there's a trap: If you find $H \setminus xy$ is nonplanar for one particular edge xy, nothing follows about $\mathrm{cr}(H)$. [-1 pt. if you claimed to prove $\mathrm{cr}(H) > 1$ that way.] If it were nonplanar for *every* edge xy you could conclude $\mathrm{cr}(H) > 1$, but unfortunately, $H \setminus ae$ is planar, so that method can't solve the problem. Crossing number is hard!

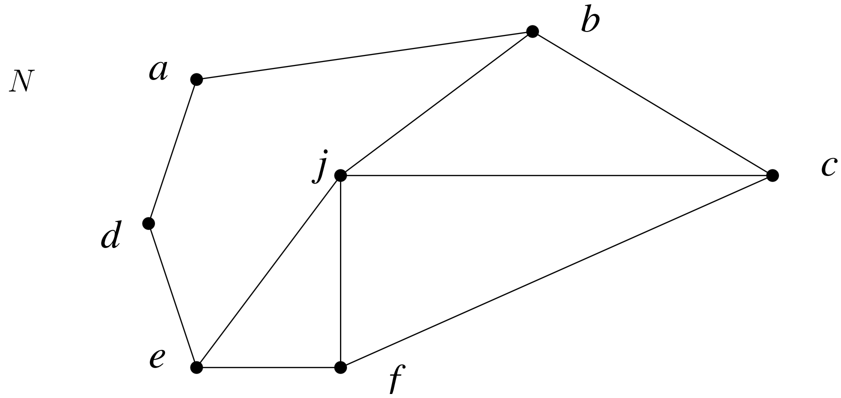
In case you're interested, here is how to prove $\mathrm{cr}(H) = 2$. I did *not* expect you to do it on the test!

Proof 1. (Advanced!) Suppose the crossing number were 1. Then there would be an edge whose deletion makes H planar. Let's check the edges. We can do several at a time by deleting vertices. Deleting any one of b, g, h, j gives a nonplanar graph. Therefore, deleting any one edge incident with b, g, h, or j gives a nonplanar graph. The only other edges are ae and cd. If $\mathrm{cr}(H) = 1$, there must be a drawing where ae and cd are the crossing edges. As I said previously, $H \setminus ae$ is planar. But (it can be proved) there is only one way to draw $H \setminus e$ in the plane with no crossings (except that you can choose which region is the outer region). If you draw $H \setminus e$ without crossings, you have to put e into one of the regions. No matter which region you pick, you'll need two crossings to connect e to all its adjacent vertices (a, g, and j).

Proof 2. (Not advanced, but complicated.) This proof is based on ideas of Rick Behr. Notice the disjoint cycles $ejge$ and $abcdha$ (reading clockwise), each of which contains two neighbors of f. Consider how to draw H. If the two cycles cross they must cross twice (at least) so we can't get only 1 crossing. If one is inside the other, and we put f inside both, or outside both, the edges from f to its neighbors will create 2 crossings (at least). Therefore f has to be in the annulus (ring) between the cycles. We can put $ejge$ inside $abcdha$ (because if we do the reverse we can simply turn the diagram inside out). Now we have edges

ae, jc, gc, gd across the annulus, cutting it into four regions. No matter which one we put f in, it has at least two neighbors outside its region, which will form at least 2 crossings. But wait! what if we flip $ejge$ over, making $egje$ (still inside $abcdha$)? The edges ae, jc, gd will then have at least one crossing, and it's easy to check that no matter where we put f one or more of its edges will cross another edge. Therefore, no drawing of H can have fewer than two crossings.
But wait! What if we gave the cycle $abcdha$ a twist, so it crosses itself once? We have to show there must be at least one more crossing. I leave that to you (or someone else).

10. [Points: 5] Find the dual graph of the plane graph N.



Solution: I want to see that you know what a planar dual graph is. It's not necessarily a "graph"; in this case it's a pseudograph. Don't forget the outer region, which gives one vertex of the dual and has 3 edges to the region bounded by $abjeda$.

11. [Points: 7+7]
   (a) A plane drawing of a connected cubic graph G_1 has 40 regions. How many vertices does G_1 have?
   (b) A plane drawing of a maximal planar graph G_2 has $r = 44$ regions. What is the average degree of G_2?

Solution: (a) As G_1 is cubic, we can calculate $q = \frac{3}{2}p$. As G_1 is connected, we can apply Euler's formula: $2 = p - q + r = p - \frac{3}{2}p + 40$, so $p = 2 \cdot 38 = 76$.

Key points: Euler's formula. Relation between p and q for a cubic graph (which you can calculate quickly if you're not sure you remember).

Solution: (b) Since the graph is maximal planar, every region is a triangle; so $3r = 2q$. That implies $q = 66$. Since a maximal planar graph must be connected, Euler's formula applies; therefore $2 = p-q+r = p-66+44$, so $p = 24$. The average degree is $2q/p = 11/2$.

Another proof: Use $q = 3(p-2)$, true for a maximal planar graph, together with Euler's formula, to get q and p.

Key points: A maximal planar graph is connected. Stating the reason Euler's formula applies to a maximal planar graph. Every region is a triangle (N.B. that's not the definition!). Formula for average degree.