Automorphisms of Graphs
Math 381 — Spring 2011

An *automorphism* of a graph is an isomorphism with itself. That means it is a bijection, $\alpha : V(G) \to V(G)$, such that

$$\alpha(u)\alpha(v) \text{ is an edge if and only if } uv \text{ is an edge.}$$

We say α *preserves edges and non-edges*, or as the book says, it *preserves adjacency and nonadjacency*. (If you took combinatorics, you'll remember that a bijection from a set to itself is called a *permutation* of that set; so an automorphism is a permutation of the vertex set, but generally speaking not every permutation will be an automorphism.)

Every graph has the *trivial automorphism* $\mathrm{id} : V \to V$ defined by $\mathrm{id}(v) = v$. (In other words, it does nothing.) Most graphs (whatever that means exactly, since there are infinitely many finite graphs!) have no other automorphisms, but many interesting graphs have many automorphisms. I discussed the Petersen graph P in class. It has, for its size, a huge number of automorphisms.

We write $\operatorname{Aut} G$ for the set of all automorphisms of G. Thus, $\mathrm{id} \in \operatorname{Aut} G$ for every graph.

Example Aut.1. Here is a graph N that has no nontrivial automorphisms.

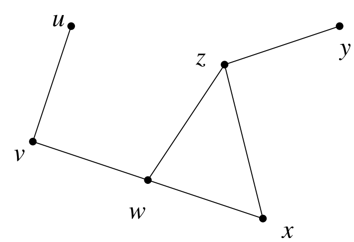

Figure Aut.1. A graph N for which $|\operatorname{Aut}(N)| = 1$.

Let's prove that $\operatorname{Aut} N = \{\mathrm{id}\}$, that is, N has no nontrivial automorphisms. Suppose α is an automorphism. Like any isomorphism, $\alpha(v_0)$ must have the same degree as v_0, for any vertex v_0. So, in particular, $\alpha(u) = u$ or y. The neighbor of u is v, so $\alpha(v)$ must be a neighbor of $\alpha(u)$. That means $\alpha(v) = z$ if $\alpha(u) = y$, but then v and $\alpha(v)$ don't have the same degree, which is impossible for an automorphism. Therefore, $\alpha(u) = u$; that implies $\alpha(v) = v$ and also $\alpha(y) = y$ since $\alpha(y)$ cannot be the same as $\alpha(u)$ (remember that an automorphism is a bijection). Now $\alpha(w)$ has to be a neighbor of $\alpha(v) = v$ and cannot be u, so $\alpha(w) = w$. The only remaining vertex is x, and $\alpha(x)$ can't be anything already in the image of α, which leaves only x as a possible value for $\alpha(x)$. We've proved that $\alpha(v_0) = v_0$ for every vertex, so $\alpha = \mathrm{id}$. Therefore, the only automorphism of N is id.

Example Aut.2. C_4 is the opposite: it has quite a few automorphisms, in fact 8. Number the vertices v_1, v_2, v_3, v_4 in order around C_4. (See any graph in Figure Aut.2.) It's convenient to define $Z_4 := \{1, 2, 3, 4\}$.

Let's examine the possible automorphisms α of C_4. First of all, $\alpha(v_1) = v_i$ for some $i \in Z_4$. Then $\alpha(v_2)$ has to be a neighbor of v_i, so it is either v_{i+1} or v_{i-1}. (We take the subscripts modulo 4, which means that if $i + 1 > 4$, we subtract 4 so it is in Z_4, and if $i - 1 < 1$, we add 4 to put it in Z_4.) Now $\alpha(v_4)$ has to be the neighbor of v_i that isn't $\alpha(v_2)$, so it is v_{i-1}

or v_{i+1}, whichever one is not $\alpha(v_2)$. Finally, v_{i+2} is the only remaining vertex not yet in the image of α, so it must be equal to $\alpha(v_2)$. We can display α in a table:

$v =$	v_1	v_2	v_3	v_4
$\alpha(v) =$	v_i	$v_{i\pm1}$	$v_{i\mp1}$	v_{i+2}

(In reading the table, note that $i \mp 1$ is the opposite of $i \pm 1$, that is, $\mp = -$ if $\pm = +$ and $= +$ if $\pm = -$. Also, since we take numbers modulo 4, $i+2$ and $i-2$ give the same result, and $i \mp 1$ is the same as $i \pm 3$. This is called "modular arithmetic".)

You'll notice that we had 4 choices for i, and then we had to choose either $i+1$ or $i-1$, which is 2 further choices. That gives us $4 \times 2 = 8$ ways to choose an automorphism of C_4; thus, $|\operatorname{Aut}(C_4)| = 8$, as claimed.

An example is α in Figure Aut.2, where $i = 3$ and $\pm = +$ (that is, $i \pm 1 = i+1$). β is another example; there $i = 1$ and $\pm = -$.

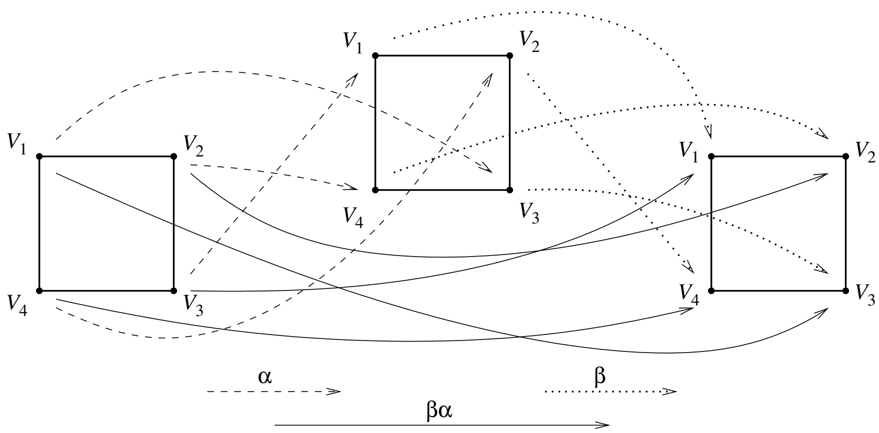

FIGURE AUT.2. C_4 and automorphisms α, β, and their product $\beta\alpha$.

We'll discuss the product $\beta\alpha$ in Example Aut.3.

The simplest examples are K_n and $\bar{K}_n$. Every permutation of the vertex set is an automorphism.

Exercise Aut.1. Prove that $\operatorname{Aut} G$ consists of all permutations of V if and only if G is complete or has no edges.

Exercise Aut.2. Find $\operatorname{Aut} H$; that is, find all automorphisms of H in Figure Aut.3..

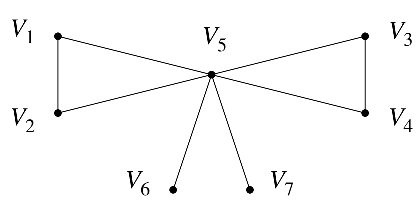

FIGURE AUT.3. A cute graph H.

As an automorphism is a function, two automorphisms can be composed. Suppose α and β are automorphisms of G. Then they are both bijections $V \to V$. As you know, the composition of two bijections is a bijection. Therefore, $\beta\alpha$ (the rule is: do α first, then β;

the technical definition is $\beta\alpha(v) = \beta(\alpha(v)))$ is a bijection $V \to V$. But that doesn't mean it is an automorphism. So, there's a question: Is it?

Similarly, a permutation of V, α, has an inverse function α^{-1} (because it is a bijection). Is α^{-1} an automorphism?

Theorem Aut.1. *The composition of two automorphisms of G is an automorphism of G. The inverse of an automorphism of G is also an automorphism.*

Proof. I will prove the first statement. The second is an exercise.

The definition of an automorphism of G is that it is a bijection $f : V \to V$ such that $f(u)f(v) \in E \iff uv \in E$. We want to prove this for $f = \beta\alpha$. We know (from Math 330 or equivalent) that it's a bijection. We need to know it preserves edges and non-edges.

We know that's true for α and for β. Now, let's write the proof using what we know.

We know α is an automorphism; that means

$$uv \in E \iff \alpha(u)\alpha(v) \in E.$$

We also know β is an automorphism; that means

$$u'v' \in E \iff \beta(u')\beta(v') \in E.$$

In particular, that holds true if we set $u' = \alpha(u)$ and $v' = \alpha(v)$; so we can write

$$\alpha(u)\alpha(v) \in E \iff \beta(\alpha(u))\beta(\alpha(v)) \in E.$$

Now we combine both if-and-only-ifs:

$$uv \in E \iff \alpha(u)\alpha(v) \in E \iff \beta(\alpha(u))\beta(\alpha(v)) \in E.$$

The conclusion is that

$$uv \in E \iff \beta\alpha(u)\beta\alpha(v) \in E.$$

This is the definition of $\beta\alpha$'s being an automorphism. □

Exercise Aut.3. Prove the second statement of Theorem Aut.1.

This means we can "multiply" two automorphisms of G, and we can invert an automorphism. A system in which you can "multiply" any two elements (and they satisfy the associative law) and invert any element is called a *group.* (Students of modern algebra will know this.) Thus, Theorem Aut.1 is saying that $\operatorname{Aut} G$ is a group. It is called the *automorphism group* of G. There has been and still is a great deal of research done about it.

Example Aut.3. Figure Aut.2 shows two automorphisms, α and β, and the product $\beta\alpha$. Here is a table showing how to compute the product:

$v =$	v_1	v_2	v_3	v_4
$\alpha(v) =$	v_3	v_4	v_1	v_2
$\beta(v) =$	v_1	v_4	v_3	v_2
$\beta(\alpha(v)) =$	v_3	v_2	v_1	v_4

Exercise Aut.4. The octahedral graph O (Figure Aut.4) has an automorphism α such that $\alpha(v_i) = v_{i+1}$ in the numbering of the figure,[1] and another automorphism β such that

[1] If you get an index $i+1 > 6$, reduce modulo 6; that is, subtract 6 so your number is in the range $1, \ldots, 6$.

$\beta(v_i) = v_{i+3}$ for $i = 1, 2, 3$ and $\beta(v_i) = v_{i-3}$ for $i = 4, 5, 6$.[2] Calculate the exact values of $\alpha\beta(v_i)$ and $\alpha^{-1}(v_i)$ for $i = 1, \ldots, 6$.

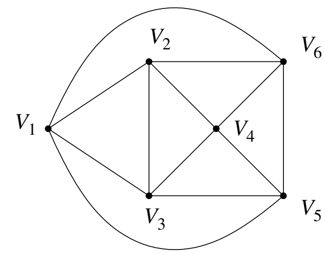

FIGURE AUT.4. Octahedron graph.

Exercise Aut.5. (a) Find Aut C_n. That is, find all automorphisms of C_n. Warning: Aut C_3 is possibly misleading; it is not typical.
(b) Find Aut $\bar{C}_n$.

Exercise Aut.6. (a) Find Aut P_n, $n \geq 1$.
(b) Find Aut $\bar{P}_n$.

Exercise Aut.7. (a) Find Aut O.
(b) Find Aut $\bar{O}$.

Exercise Aut.8. (a) Find Aut $K_{m,n}$.
(b) Find Aut $\bar{K}_{mn}$.

Exercise Aut.9. How are Aut G and Aut $\bar{G}$ related?

The Petersen Graph. The Petersen graph P is a remarkable example because it has more automorphisms, relative to its size, than almost any other graph. We can see why by looking at a systematic construction of P. Let $Z_5 := \{1, 2, 3, 4, 5\}$ and let $\mathcal{P}_2(Z_5)$ be the set of all unordered pairs of elements of Z_5. Then $V(P) = \{v_{ij} : \{i, j\} \in \mathcal{P}_2(Z_5)\}$; in words, we label the vertices of P by the unordered pairs of numbers from 1 to 5. We think of v_{ji} and v_{ij} as identical; for instance, v_{12} and v_{21} are the same vertex Two vertices in P are adjacent if and only if their labels are disjoint sets; formally, $E(P) = \{v_{ij}v_{kl} : \{i, j\} \cap \{k, l\} = \varnothing\}$.

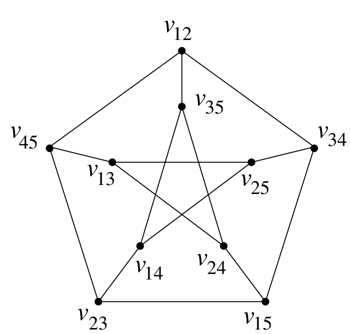

FIGURE AUT.5. The Petersen graph with the 2-index vertex labelling.

[2]If you work modulo 6, then $\beta(v_i) = v_{i+3}$ for all i.

If we permute the elements of Z_5 we get a bijection $V(P) \to V(P)$. For instance, suppose we have a permutation ζ of Z_5 such that $\zeta(1) = 2$, $\zeta(2) = 1$, $\zeta(3) = 3$, $\zeta(4) = 5$, and $\zeta(5) = 4$. Then ζ gives us the automorphism α of the following table:

Vertex v	v_{12}	v_{13}	v_{14}	v_{15}	v_{23}	v_{24}	v_{25}	v_{34}	v_{35}	v_{45}
$\alpha(v)$	v_{12}	v_{23}	v_{25}	v_{24}	v_{13}	v_{15}	v_{14}	v_{35}	v_{34}	v_{45}

Any permutation of Z_5 gives an automorphism of P in the same way, and different permutations give different automorphisms. (See Exercise Aut.11; it's fairly easy.) Since there are $5! = 120$ permutations of Z_5, there are at least 120 automorphisms of P. It's possible to prove there are no more than 120, but I'll omit that. Summarizing:

$$|\operatorname{Aut}(P)| = 120.$$

Exercise Aut.10. What does this description of P tell you about how it's related to line graphs?

Exercise Aut.11. This is about how the permutations of Z_5 are related to the automorphisms of P.

(a) Prove that any permutation of Z_5 gives an automorphism of P
(b) Prove that different permutations of Z_5 give different automorphisms of P.
(c) (Optional.) Prove there are no other automorphisms of P. (Not easy, unless you know some theory of permutation groups.)

Example Aut.4. Let's try an example to illustrate the theorem of Exercise Aut.11(c). I'll produce an automorphism α of P by a simple redrawing of the usual picture, and deduce the permutation of Z_5 that gives that automorphism.

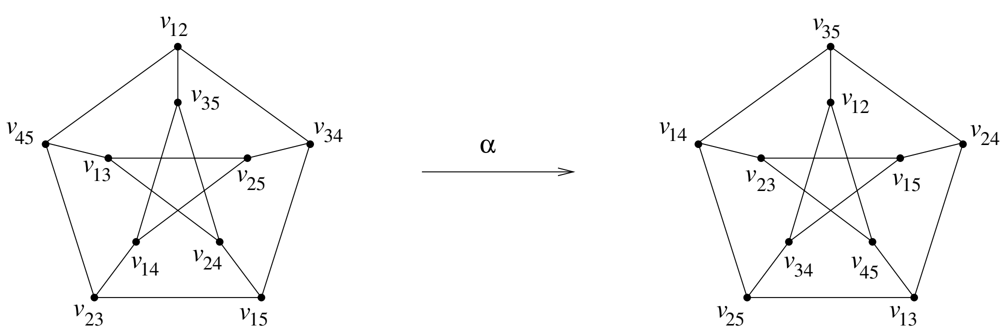

FIGURE AUT.6. An automorphism of P produced by reversing the outer and inner pentagons.

As you can see, just redrawing the inner (star) pentagon as an outer pentagon and the outer as the inner pentagon makes a new picture that looks exactly like the original; thus, it's clearly an automorphism. Here is the automorphism as a bijection $V \to V$:

Vertex v	v_{12}	v_{13}	v_{14}	v_{15}	v_{23}	v_{24}	v_{25}	v_{34}	v_{35}	v_{45}
$\alpha(v)$	v_{35}	v_{23}	v_{34}	v_{13}	v_{25}	v_{45}	v_{15}	v_{24}	v_{12}	v_{14}

What could be the permutation ζ of Z_5 that gives the automorphism α? (Its existence is guaranteed by Exercise Aut.11(c).) We track the subscripts. $\alpha(v_{12}) = v_{35}$ means that $1, 2$ become $3, 5$; so either $1 \to 3,\ 2 \to 5$ or $1 \to 5,\ 2 \to 3$. We can't tell which, yet, but we'll look at another subscript. Since $\alpha(v_{15}) = v_{13}$, either $1 \to 1,\ 5 \to 3$ or $1 \to 3,\ 5 \to 1$. Consistency requires that ζ take $1 \to 3$; therefore, $2 \to 5$ and $5 \to 1$. We don't know what ζ does with 3 or 4 yet, so we need to look at another vertex. Let's try v_{23}; as $\alpha(v_{23}) = v_{25}$, and as we already found that $2 \to 5$, ζ must carry $3 \to 2$. Then $4 \to 4$, since we can't carry 4 to anything that's already in the image of ζ. Summarizing ζ, it is

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 5 & 2 & 4 & 1 \end{array}$$