The Matrix-Tree Theorem
Math 381
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Chapter 5 gives formulas for the number of spanning trees in some graphs. There is a remarkable formula for the number of spanning trees in any graph. It involves a matrix associated with a graph.

Definition MT.1. The *adjacency matrix* of a graph G with vertex set $V = \{v_1, v_2, \ldots, v_p\}$ is the $p \times p$ matrix $A(G) = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0 & \text{if } v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

In particular, the diagonal is all 0, since v_i is not adjacent to itself in a graph.

The adjacency matrix is a symmetric matrix; therefore, by the diagonalization theorem of symmetric matrices, it has all real eigenvalues. We won't go into eigenvalues of adjacency matrices, but there is a lot of research on them right up to now.

Definition MT.2. The *degree matrix* of G is the $p \times p$ matrix $D(G)$ with the degree of v_i on the diagonal in row and column i, and with 0's off the diagonal.

The next matrix is the one we are really interested in.

Definition MT.3. The *Kirchhoff matrix* of G is the matrix $K(G) = D(G) - A(G)$.

Example MT.1. Let $G = K_4 \setminus v_2v_4$. Then

$$A(G) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad D(G) = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad K(G) = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}.$$

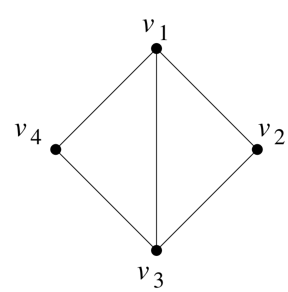

Figure MT.1. $K_4 \setminus v_2v_4$

Theorem MT.1. *The determinant of the Kirchhoff matrix is zero:* $\det K(G) = 0$.

Proof. The sum of the entries in row i of the adjacency matrix $A(G)$ is $\deg v_i$, by the definition of degree. Therefore, in $K(G)$, the sum of the entries in row i is $-\deg v_i + \deg v_i = 0$. $\square$

This is not why the Kirchhoff matrix is interesting. Let's delete one row and one column from it and *then* take the determinant. We call this matrix K_{ij}, if row i and column j are deleted. For example:

Example MT.2. Apply that procedure to Example MT.1. Let's delete row 1 and column 1.

$$K(G)_{11} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad \det K(G)_{11} = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{vmatrix} = (12+0+0)-(2+2+0) = 8.$$

Another way to do it:

$$K(G)_{13} = \begin{pmatrix} -1 & 2 & 0 \\ -1 & -1 & -1 \\ -1 & 0 & 2 \end{pmatrix}, \quad \det K(G)_{13} = \begin{vmatrix} -1 & 2 & 0 \\ -1 & -1 & -1 \\ -1 & 0 & 2 \end{vmatrix} = (2+2+0)-(0-4+0) = 8.$$

Now count the spanning trees in G. Do you get 8?

Theorem MT.2. *For any $i, j \in \{1, 2, \dots, p\}$, the value of $(-1)^{i+j} \det K(G)_{ij}$ is the number of spanning trees in G.*

I will not prove this theorem. It involves factoring the Kirchhoff matrix into a product $H(G)H(G)^T$ (where $H(G)$ is a matrix called the *incidence matrix* of G), a matrix theorem called the Cauchy–Binet Theorem that gives a formula for the determinant of a product of the form HH^T (where H is any matrix), and some clever analysis of graph matrices.

Exercise MT.1. (a) Find the adjacency and Kirchhoff matrices of K_3.

(b) Try several combinations of row and column deletions and find their determinants (with the sign factor), i.e., $(-1)^{i+j} \det K(K_3)_{ij}$ for several combinations of i and j. Try at least one case where $i = j$, and at least one where $i \neq j$. You notice that you always get the same value of the (signed) determinant.

(c) Count the spanning trees of K_3 directly in the graph. Compare with (b). They ought to be equal; if not, did you make a mistake?

Exercise MT.2. (a) Find the adjacency and Kirchhoff matrices of C_4, the cycle of length 4.

(b) Try several combinations of row and column deletions and find their determinants, i.e., $(-1)^{i+j} \det K(C_4)_{ij}$ for several combinations of i and j. Try at least one case where $i = j$, and at least one where $i \neq j$. You notice that you always get the same value of the (signed) determinant.

(c) Count the spanning trees of C_4 directly in the graph. Compare with (b).

They ought to be equal; if not, did you make a mistake? The most likely mistake is to evaluate a 4×4 determinant incorrectly. Make sure you know how to do it.

Exercise MT.3. Do the same for K_4.

(a) Find $A(K_4)$, $D(K_4)$, and $K(K_4)$.

(b) Write out $K(K_4)_{ii}$ for one choice of i, and evaluate its determinant by using row and column operations to simplify the matrix.

(c) Does your determinant agree with Cayley's formula $s(K_4) = 16$? If not, did you make a mistake?

Exercise MT.4. Do the same for K_n, in the following way:

(a) Find $A(K_n)$, $D(K_n)$, and $K(K_n)$.

(b) Write out $K(K_n)_{ii}$ for one choice of i, and evaluate its determinant by using row and column operations to simplify the matrix.

(c) Does your determinant agree with Cayley's formula $s(K_n) = n^{n-2}$ (Theorem 5.2.1)?

If you made no mistake, you have a proof of Cayley's formula by matrix theory.

Exercise MT.5. Do parts (a, b, c) from Exercise MT.4 for the complete bipartite graph $K_{m,n}$.

Does your result agree with Theorems 5.3.1 ($m = 2$) and 5.3.2 ($m = 3$)?