The Chromatic Polynomial
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Given a set of k colors, there is a certain number of ways to color a graph G with those colors. If k is too small , that number is 0 (because you don't have enough colors to color G). If k is large enough, that number is a positive number. So we can define a function on positive integers, which I call χ_G, by letting $\chi_G(k) =$ the number of colorings of G with k colors. Let's call this function the *chromatic function of* G. (Don't confuse χ_G, the function, with $\chi(G)$, the chromatic number.)

This function has the property that it equals 0 for every $k < \chi(G)$ and is positive for every $k \geq \chi(G)$, because if you can color a graph with k colors, you can also color it with any larger set of colors. (Remember, it's not necessary to use all the colors.)

Look at examples. You have k colors. Consider complete graphs, where all vertices are adjacent so no two can have the same color.

Ex. 1 For the tiny complete graph K_1 you only pick one color. There is one way to do that, so $\chi_{K_1}(k) = k$.

Ex. 2 For the very small complete graph K_2, say the vertices are v_1, v_2. You pick one color for v_1; there are k ways to do that. Then you pick a different color for v_2; there are $k-1$ ways to do that (for each choice of color for v_1) because you used one color. The number of ways to do this process is $k \cdot (k-1)$, so $\chi_{K_1}(k) = k(k-1)$.

Ex. 3 For the small complete graph K_3, say the vertices are v_1, v_2, v_3. First choose a color for v_1: there are k ways to choose it. Now there are $k-1$ colors left, from which you choose one for v_2; there are $k-1$ choices for this color. Then you choose one of the $k-2$ remaining colors for v_3. The total number of ways to choose the three colors is $k \cdot (k-1) \cdot (k-2)$, so $\chi_{K_1}(k) = k(k-1)(k-2)$.

Ex. 4 Let's do any K_p. For the first vertex (it doesn't matter which vertex this is) we choose from k colors. For the next vertex we have $k-1$ colors to choose from. For the third vertex we have $k-2$ colors to choose from. Etc. For the p-th vertex, we used $p-1$ colors so we have $k-(p-1) = k-p+1$ colors to choose from. The total number of ways to color the p vertices is the product, $k(k-1)(k-2)\cdots(k-[p-1])$, so this equals $\chi_{K_p}(k)$.

But suppose we don't have enough colors, i.e., $k < p$? The formula should give 0. E.g., if we have $p-1$ colors, there is no way to color K_p. But that's okay: our formula has the factor $k-[p-1] = 0$, which is the number of colorings with $p-1$ colors. Similarly, for any number of colors $m < p$, there is a factor $k-m$ in $\chi_{K_p}(k) = k(k-1)(k-2)\cdots(k-[p-1])$; then with $k = m$ colors we have the factor $k-m = 0$ so we get the right answer, $\chi_{K_p}(m) = 0$, from our formula.

Every chromatic function $\chi_{K_p}(k)$ is a polynomial of degree p and it is *monic* (the leading coefficient is 1). Here is the surprising fact:

Theorem 1. *Let G be any graph with p vertices. The chromatic function of G is a polynomial of degree p and is monic.*

So we change the name of the function and call it the *chromatic polynomial of* G. Just to explain: I'm saying there is a polynomial, $\chi_G(x)$, such that for each positive integer k, $\chi_G(k)$ is the number of ways to color G in k colors. This is not an obvious fact.

The chromatic polynomial turns out to have applications in geometry and in physics, but I will ignore that.

A theorem should have a proof, so here it is.

Proof. Let's define another function: $\psi_G(k) =$ the number of ways to color G using every one of the k colors. This is a very different function from the chromatic polynomial. $\psi_G(k) = 0$ if k is too small (specifically, $k < \chi(G)$) and also if k is too large ($k > p$, because there are too many colors to use them all).

We can compute the chromatic polynomial from the numbers $\psi_G(1), \psi_G(2), \ldots, \psi_G(p)$ as follows: Suppose we have k colors available. We can pick the number of colors to use, say m (where $m \leq k$) and then use all those m colors to color G. There are $\binom{k}{m}$ ways to choose the m colors out of our k available colors. For each choice of the m colors, there are $\psi_G(m)$ ways to color G using those m colors. So the total number of ways to color G using exactly m colors from our set of k colors is $\psi_G(m)\binom{k}{m}$. But we could have picked any value of m from 1 to p, so we should sum them up to get the total number of ways to color G with our k available colors, i.e.,

$$\chi_G(k) = \sum_{m=1}^{p} \psi_G(m) \binom{k}{m}.$$

Now we do a little algebra. You probably know that $\binom{k}{m} = \frac{k!}{m!(k-m)!}$. You may have seen that this equals $\frac{k(k-1)\cdots(k-m+1)}{m!}$. Let's write this out:

$$\binom{k}{m} = \frac{k!}{m!(k-m)!} = \frac{k(k-1)\cdots(k-m+1)(k-m)(k-m-1)\cdots(2)(1)}{m!(k-m)(k-m-1)\cdots(2)(1)}$$
$$= \frac{k(k-1)\cdots(k-m+1)}{m!}$$

by cancelling common factors in the numerator and denominator.

So now I can write

$$\chi_G(k) = \sum_{m=1}^{p} \psi_G(m) \frac{k(k-1)\cdots(k-m+1)}{m!} = \sum_{m=1}^{p} \frac{\psi_G(m)}{m!} k(k-1)\cdots(k-m+1).$$

Notice that $k(k-1)\cdots(k-m+1)$ is a monic polynomial in k of degree m. So we are adding up polynomials of degrees $m = 1, 2, \ldots, p$ with coefficients $\psi_G(m)/m!$. The highest degree is p, so our sum is a polynomial of degree at most p.

The term k^p arises only from $m = p$ and the coefficient is $\psi_G(p)/p!$. I claim that $\psi_G(p) = p!$. That is because to color using exactly p colors, we must give every vertex a separate one of the p colors, and there are $p!$ ways to do that. So the coefficient of k^p is 1. That means the highest-degree term in the polynomial $\chi_G(k)$ is $1k^p$, i.e., we have a monic polynomial of degree p. Done! □

I didn't have to know the values of the numbers $\psi_G(m)$. All I need to know is that they don't depend on k, the actual number of colors. In fact, most of the numbers $\psi_G(m)$ are virtually impossible to calculate. We need a different way to find the chromatic polynomial. A method exists; it uses deletion and contraction of one edge at a time. This will come later, if time allows.