FINAL EXAM—MATH 381—SPRING 2021
MAY 19–26, 2021
GRADING GUIDE

The 190 exam points, not counting bonus points, will be scaled to 150 course points.

(1) (10 points) Let $G$ be a connected graph. Prove $G$ has a bridge if and only if not every edge is in a cycle. Use the book's definition of a bridge (page 44).

**Solution.** I will prove that an edge is a bridge if and only if it is not in any cycle. That's essentially the same question. Note that there are two things to prove here.

(I) If $e$ is a bridge, it is not in any cycle. Equivalently (by contrapositive), if $e$ is in a cycle, it is not a bridge.

(II) If $e$ is not in a cycle, it is a bridge. Equivalently (by contrapositive), if $e$ is not a bridge, it is in a cycle.

Proof of (I). Suppose $e = uv$ is a bridge in $G$. Then there is no $uv$-path in $G - e$, because $G - e$ is disconnected. [N.B. A very rigorous proof would prove this, but I don't expect it for full credit. See (Ia).] Now suppose for contradiction that $e$ is in a cycle $C$. Then $C - e$ is a path between $u$ and $v$, giving us a contradiction. Therefore, a bridge cannot belong to a cycle.

Proof of (II). Suppose $e = uv$ is not a bridge. I will prove it belongs to a cycle. If $e$ is not a bridge, then $G - e$ is connected, so (by the definition of connection) there is a $uv$-path in $G - e$. That path together with $e$ makes a cycle in $G$ that contains $e$.

(Ia) (This will show you how much we need for a really solid proof.) For the missing part, we have to prove there is no $uv$-path in $G - e$. Suppose there were such a path, call it $P$. We prove $G - e$ would be connected using the definition of connection. We want to prove that for any two vertices, $x$ and $y$, there is an $xy$-path in $G - e$. We know there is an $xy$-path $P_{xy}$ in $G$, since $G$ is connected. If $e$ is not in $P_{xy}$, then $x$ and $y$ are connected by a path in $G - e$. If $e$ is in $P_{xy}$, then (by labeling $x, y$ suitably) we can assume $P_{xy}$ consists of a path $P_{xu}$ from $x$ to $u$, then edge $e$, then a path $P_{vy}$ from $v$ to $y$. Now, replace $e$ in this path by $P_{uv}$, so we get a walk (not necessarily a path) $W_{xy}$ from $x$ to $y$ that consists of $P_{xu}$, then $P_{uv}$, then $P_{vy}$. Now, by the Connection Theorem on the announcements page, the existence of $W_{xy}$ in $G - e$ implies the existence of an $xy$-path $Q_{xy}$ in $G - e$. We have proved that every pair of vertices in $G - e$ is joined by a path. That is the definition of $G - e$ being connected. But that contradicts the assumption that $e$ is a bridge. Therefore, no $uv$-path exists in $G - e$, which means there cannot be a cycle $C$ in $G$ that contains $e$.

**Grading rubric.**

*5 pts.* (I)

*3 pts.* How does the cycle enter into this reasoning?

*2 pts.* "Any edge in a cycle of $G$ is not a bridge." It is not a bridge of the cycle, but that doesn't prove it is not a bridge of $G$.

*0 pts.* Vagueness.

*5 pts.* (II)

*1 pts.* Pick $e = uv$ that is not in a cycle. Because $e$ is not in a cycle, there is no $uv$-path in $G - e$. [Why not? That is the key point.] Thus, $G - e$ is disconnected so $e$ is a bridge.

*0 pts.* Vagueness.

(2) (10 points) A graph $R$ is $2k$-regular and has no 1-factor (where $k$ is an integer $\geq 2$). Prove that its edge chromatic number is $2k+1$.

**Solution.** Step 1. Suppose by way of contradiction that $R$ is edge $2k$-colorable. Color the edges in $2k$ colors. Each vertex has $2k$ incident edges, so there must be one of each color. Therefore, the set of edges of color 1 must have degree 1 at every vertex of $R$, and the same for all the other colors. That is, the set of edges of any one color is a 1-factor. But I said the poor graph $R$ lacks a 1-factor, so we have a contradiction. It follows that $\chi'(R) > 2k$.

Alternative Step 1. Suppose by way of contradiction there is an edge coloring in $2k$ edges. $q = \frac{1}{2}(p \cdot 2k) = pk$ so the average number of edges per color is $q/2k = p/2$. There must be a color with at least the average number of edges. A color with $p/2$ edges uses all $p$ vertices, so it can't have more edges; therefore, it uses all $p$ vertices. That means the edges of this color are a 1-factor. But there is no 1-factor. So there can't be any coloring using $2k$ colors.

Step 2. By Vizing's theorem, $\chi'(R) = 2k$ or $2k+1$, so we conclude it equals $2k+1$.

[There is nothing special about even degree $2k$. I could have said $r$-regular where $r \geq 2$, allowing odd degree; the same proof works.]

**Grading rubric.**

*6 pts.* for not explaining exactly how a 1-factor is involved.
*2 pts.* for stating Vizing,
$-2$ *pts.* for only one graph.
*2 pts.* for complete graphs only.
$-2$ *pts.* for using turning trick.

(3) (10 points) Show that the line graph of a connected cubic graph $G$ is conservative.

**Solution.** (I made a mistake here.)

The line graph $L(G)$ of $G$ is connected and 4-regular. Theorem 3.1.4 tells us it decomposes into two 2-factors. If the 2-factors are each a single cycle, Theorem 6.2.1 tells us the line graph is conservative. If not, however, then there is no theorem that can be applied.

**Grading rubric.**

*10 pts.* Noticing all those observations.
*Bonus pts.* if you go beyond this solution.
*8 pts.* if you assume the 2-factors in Theorem 3.1.4 are Hamiltonian.
*4 pts.* if you don't use 3.1.4 or a similar valid reason.
*0 pts.* Confused "conservative" with "connected".
*4 pts.* for using 6.2.1 only on specific graphs.

(4) (5 points) Is there a magic labeling of $K_8$? Don't use any exercise in the book. Hint: First find the magic sum.

**Solution.** The magic sum for $K_n$ would be $s = \frac{1}{n}(\text{sum of vertex labels}) = \frac{1}{n} \cdot 2 \cdot (\text{sum of edge labels}) = \frac{2}{n}(1+2+\cdots+q) = \frac{2}{n}\frac{q(q-1)}{2}$. For $K_8$ this is $s = \frac{1}{8} \cdot 29 \cdot 28 = 812/8 = 101.5$. The magic sum has to be an integer, so there is no such magic labeling.

(Magic squares are irrelevant.)

**Grading rubric.**

*4 pts.* for error like dividing by degree, but still disproved.

*2 pts.* Summing the edge labels correctly,
*5 pts.* Forgetting to double label sum.

(5) (3 points) Find an antimagic labeling of $K_4$.

**Solution.** One solution (of many): Label nonadjacent pairs of edges as follows: 1, 4 for the first pair, 2, 5 for the second, 3, 6 for the third.

**Grading rubric.**
*0 pts.* Not using edge labels $1, \ldots, 6$.
*0 pts.* Graph is not $K_4$.

(6) (5 + 5 bonus points) Find a largest graph (the most edges) on 15 vertices, that has no even cycles. (There are many; you only need to find one.) For bonus credit give a proof.

**Solution.** $t$ triangles attached to each other at single vertices in a connected graph while making no other cycles (thus, in a tree-like manner) will use $2t + 1$ vertices. You should have 7 triangles and 21 edges.

This is the only way to maximize the number of edges. If you have a longer cycle instead of a triangle, you'll have fewer edges. If you have a cycle and additional edges connecting vertices of the cycle, you will have an even cycle in there. If the triangles are not connected in a tree-like pattern, you'll have those additional edgers, so an even cycle.

(There is different, very nice proof starting with a spanning tree of the 15 vertices and adding edges so as to maximize the number of cycles while not creating even cycles.)

**Grading rubric.**
*2 bonus pts.* for proof attempt using examples.
*1 bonus pts.* for noting why no edges can be added to your example. (Generously allowed.)
*3 bonus pts.* for incomplete proof attempt using good general reasoning.
*4 pts.* for 20 edges.
*3 pts.* for 19 edges.
*0 pts.* if an even cycle.

(7) (35 points) For the labeled Petersen graph $P$ (see my drawing; use the labels), answer the following questions.

(a) (10 points) Does it have a Hamilton cycle?
**Solution.** No. There are several proofs, of which I present three and a half, partly drawn from students' solutions.

*Direct proof 1:* Try to construct a Hamilton cycle $H$. We know it must have a "spoke" edge, so let's assume $H$ contains $a_1b_1$ and $a_1a_2$. (By symmetry of $P$, this covers all possible cases.) That rules out $a_1a_5$, which means $H$ must contain $a_5b_5$ and $a_5a_4$, since $a_5$ is in $H$. Now, $H$ must contain either $a_4a_3$ or $a_2a_3$, and by the symmetry of both $P$ and the partial $H$ we alrealdy have, these are not essentially different, so let's say $H$ contains $a_2a_3$. Then it cannot contain $a_2b_2$, so it must contain $b_2b_4$ and $b_2b_5$ in order to pass through $b_2$. Now $H$ contains two edges at $b_5$ so it cannot contain $b_3b_5$; that implies the other two edges at $b_3$ are in $H$; they are $b_1b_3$ and $a_3b_3$. Oh, oh! $H$ contains a 5-cycle, $a_1a_2a_3b_3b_1a_1$. That means $H$ can't possibly be a Hamilton cycle.

*Direct proof 2:* A Hamilton cycle $H$ must contain a positive even number of spoke edges in order to connect the inner and outer pentagons. Suppose $H$ contains four spoke edges, say all but $a_1b_1$. Then it must contain the path $a_5a_1a_2$, therefore not $a_2a_3$ or $a_5a_4$. That implies it contains the path $b_3a_3a_4b_4$. Since $H$ does not contain $a_1b_1$, it must contain the path $b_4b_1b_3$, but this makes a 5-cycle $b_4b_1b_3a_3a_4b_4$ in our Hamilton cycle, which is impossible. Thus, there cannot be a Hamilton cycle.

*Direct proof 3 outline:* Suppose a cubic graph $G$ of order 10 has a Hamilton cycle $H$. Then $G$ is constructed by taking a $C_{10}$ and adding 5 edges, one at each vertex. To get $P$ we have to avoid having any 3-cycle or 4-cycle. This is impossible (I omit that part of the proof; it's not hard).

*Indirect proof, using (d):* $P$ is cubic and has edge chromatic number $\chi' = 4$ from part (d) (if you found that). By Theorem 2.3.5, $P$ has no Hamilton cycle.

**Grading rubric.**

*5 pts.* for assuming $P$ is a snark, without proof.
*1 pts.* for answer with no or negligible proof.
*5 pts.* for a significant partial proof with much missing.

(b) (10 points) Does it have an Eulerian circuit?

**Solution.** No. All degrees are odd.

**Grading rubric.**

*1 pts.* No reason.

(c) (5 points) What is its clique number (the largest size of a clique)?

**Solution.** 2. There are no triangles ($C_3 = K_3$ subgraphs).

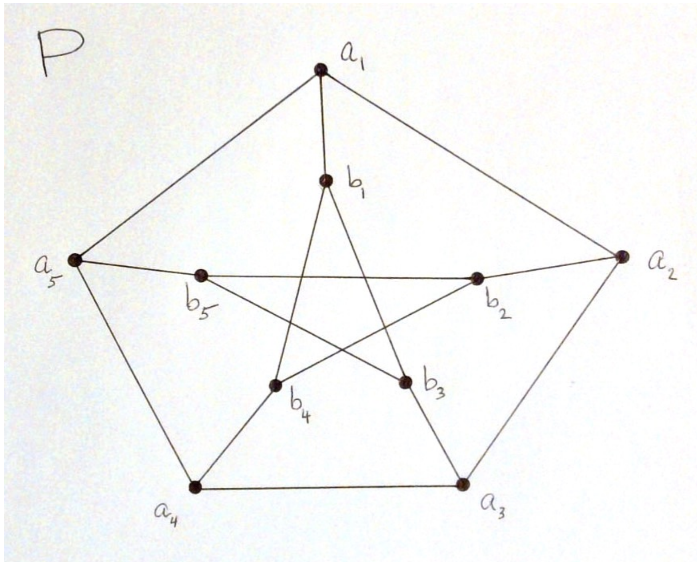
(d) (10 points) What is its edge chromatic number $\chi'(P)$?

**Solution.** *Direct proof:* $\chi'(P) \geq 3$ because $P$ is 3-regular (or because $P$ contains a $C_5$, which is not edge 2-colorable). We try to 3-color the edges of $P$. The outer pentagon has to have three colors, which can only be in one way (up to the rotational symmetry of the diagram), say color 1 on $a_1a_2$ and $a_3a_4$, color 2 on $a_2a_3$ and $a_4a_5$, and color 3 on $a_1a_5$. That forces the spoke edge colors; in particular, 1 on $a_1b_1$ and 3 on $a_3b_3$ and $a_4b_4$, which forces color 2 on $b_1b_3$ and $b_1b_4$. But wait! $b_1b_3$ and $b_1b_4$ are adjacent, so the 3-coloring has led to a contradiction. We made no arbitrary choices, so 3-coloring is impossible. By Vizing's theorem, $\chi'(P) = 4$.

*Indirect proof, using (a):* Suppose $P$ has an edge 3-coloring. Since $P$ is 3-regular, the set of edges with each color is a 1-factor. Delete these edges, leaving a 2-factor with 10 edges and 2 colors. This 2-factor is not a Hamilton cycle, by part (a), therefore it is the union of two or more cycles, but the girth is 5, so the 2-factor is a union of two 5-cycles. These 5-cycles can't be colored in 2 colors, so we have a contradiction. Therefore, $\chi'(P) > 3$. By Vizing's theorem, $\chi'(P) = 4$.

**Grading rubric.**

*2 pts.* for only $\chi'(P) \geq 3$.
*2 pts.* for only $\chi'(P) \leq 4$.
*4 pts.* for only $3 \leq \chi'(P) \leq 4$.
*6 pts.* for a significant partial proof with much missing.
*8 pts.* for most of a proof but with a significant gap.
*0 pts.* No reason.




(8) (55 points) Solve the following problems for the complement $\overline{P}$ of the labeled Petersen graph $P$ (see my drawing).

**Solution.** If you assume HW # M6 is true without proof, I accept it for this problem.

If you use the $ij$ labeling of vertices, that is okay.

For completeness, I could prove M6 by making a drawing of $\overline{L(K_5)}$ and showing that it looks just like the Petersen graph (or vice versa). This implies that $\overline{P} = L(K_5)$, so I can use $L(K_5)$ in this problem. This means we have 10 vertices $ij$ (meaning the set $\{i, j\} \subset \{1, 2, 3, 4, 5\}$), corresponding to the edges of $K_5$, and we have edges $\{ij, ik\}$ for any two $j, k \neq i$, derived from the adjacency of edges in $K_5$. I will explain a cute way to draw this in part (h).

(a) (5 points) Find the complement $\overline{P}$ of $P$. (A labeled picture will be a good answer.)

**Solution.** A readable picture.

**Grading rubric.**

$-1$ *pts.* per 1 or 2 errors.

(b) (0 points) Find the degree of every vertex in $\overline{P}$ (this is to verify your complement).

**Solution.** 6. (If you don't get 6 at every vertex, there is a mistake.)

Explanation: $9 - 3 = 6$. The 9 is the degree of the complete graph $K_{10}$ and the 3 counts the edges of $P$, which are omitted.

(c) (5 points) Find the clique number (the largest size of a clique) for $\overline{P}$.

**Solution.** The clique number $\omega(\overline{P}) = 4$. The reason is that $\overline{P} = L(K_5)$, so the largest cliques are composed of the $K_5$ edges incident with a single vertex. (The only other cliques in a line graph come from triangles in the base graph, so they give no clique larger than 3 vertices. I am not expecting you to prove that.)

Solution 2: You can find a $K_4$ subgraph, so $\omega \geq 4$. Any clique can have at most two vertices in the $a$'s and at most two in the $b$'s, so at most 4 vertices, so $\omega \leq 4$. Therefore, $\omega = 4$.

**Grading rubric.**

*2 pts.* for 3 without disproof of 4.
*4 pts.* for 4 without a pretty good attempt to disprove 5.

(d) (10 points) Does $\overline{P}$ have the most possible edges for a graph with the same clique number and the same number of vertices? (Use your clique number, even if you're not sure of it.)

**Solution.** By Turan's theorem, the most edges for a graph of order $p$ and clique number 4 (so no $K_5$ subgraph) is obtained by taking $K_{n_1,n_2,n_3,n_4}$ where $n_1 + n_2 + n_3 + n_4 = p$ and the $n_i$'s are nearly equal. That means $K_{3,3,2,2}$, which has $3 \cdot 7 + 3 \cdot 4 + 2 \cdot 2 = 37$ edges. Our 6-regular graph has $\frac{1}{2}(6p) = 30$ edges. The answer is No.

Solution 2: Find an edge you can add without changing $\omega$. Any edge will suffice, by the same reasoning as in (c2).

**Grading rubric.**

*4 pts.* Plausible but invalid method. E.g., adding a vertex to only one maximum clique with one new edge.
*0 pts.* No relevant reason.

(e) (5 points) Find the girth $g(\overline{P})$.

**Solution.** $g(\overline{P}) = 3$ by the obvious existence of triangles.

(f) (10 points) Find the chromatic number $\chi(\overline{P})$.

**Solution.** *The easy method.* The chromatic number of the line graph $L(K_5)$ is the edge chromatic number of the base graph $K_5$, which is 5 by Theorem 2.2.4.

*Direct method* (thanks to Zixiao Lin). A 4-clique implies $\chi(\overline{P}) \geq 4$. Try coloring in 4 colors. Start with a 4-clique, say vertices $a_1, a_2, b_3, b_4$ colored $1, 2, 3, 4$, respectively. Of the remaining vertices, the three are nonadjacent to $a_1$ form a triangle, so only one remaining vertex can have color 1. Similarly, only one can have color 2, likewise for color 3, likewise for color 4. That shows 4 colors do not suffice. Then produce a 5-coloring to prove $\chi(\overline{P}) \leq 5$, thus $= 5$.

**Grading rubric.**

*8 pts.* for knowing how to find $\chi(G)$, if you made a minor mistake yielding a wrong answer.
*6 pts.* for knowing how to find $\chi(G)$, if you did not cover all cases.
*5 pts.* for $4 \leq \chi \leq 5$ by coloring.
*5 pts.* for $4 \leq \chi \leq 6$ by coloring.
*5 pts.* for coloring without enough system, starting from $\omega$, getting $\chi > 4$.
$-2$ *pts.* for omitting $\chi \geq$ your $\omega$.
*2 pts.* for $4 \leq \chi \leq 10$.

(g) (10 points) Is $\overline{P}$ planar? Use any method.

**Solution.** Since $\overline{P}$ is 6-regular it has $q = \frac{1}{2}(6p) = 3p$. It cannot be planar, because every planar graph has $q \leq 3p - 6$.

Harder method (but not hard): Find a subdivision of $K_5$ or of $K_{3,3}$ in $\overline{P}$.

**Grading rubric.**

*5 pts.* for alleged but incorrect subdivision.
*0 pts.* Guess.
*2 pts.* for claiming a $K_5$ or $K_{3,3}$ subgraph (not subdivided). There is none.

(h) (10 points) Find the thickness $\theta(\overline{P})$.

**Solution.** $\theta(\overline{P}) > 1$ because the graph is nonplanar by part (g). A drawing can prove $\theta(\overline{P}) \leq 2$, therefore the thickness is 2.

*Drawing 1.* I will describe a decomposition into two planar subgraphs that doesn't require a drawing.

The first subgraph $G_4$ consists of all edges $\{ij, ik\}$ with $i, j, k < 5$. This subgraph is $L(K_4)$, which is $K_6 -$ (1-factor). It is easy to make a plane drawing.

The second subgraph $G_5$ consists of all edges $\{5i, 5j\}$ where $i, j < 5$ and edges $\{5i, ij\}$ where $i, j < 5$. The first edges give a $K_4$ subgraph with vertices $51, 52, 53, 54$ and all edges. For any two of these vertices, $5i$ and $5j$, there is a vertex $ij$ adjacent to both but not to any other vertex $5k$. So, we draw a curve from vertex $5i$ to $5j$ near the edge between them, and we put vertex $ij$ in the middle of this curve, making a triangle $5i, 5j, ij$. This graph is planar because it looks like $K_4$ with doubled edges.

Every edge of $L(K_5)$ is in $G_4$ or $G_5$ but not both. Thus, $L(K_5)$ decomposes into the two planar subgraphs $G_4$ and $G_5$.

*Drawing 2.* If the first planar subgraph is a 4-clique, the remainder can be drawn as planar overlapping triangles.

**Grading rubric.**

*10 pts.* for $2 \leq \theta \leq 2$.
*9 pts.* for $2 \leq \theta \leq 3$. (E.g., from $\theta(\overline{P}) \leq \theta(K_{10}) = 3$ by Theorem 9.2.3.)
$-n$ *pts.* for major errors in decomposing $L(K_5)$ into planar subgraphs.
*2 pts.* for $\theta \geq 2$. (So, $-2$ pts. for omitting $\theta \geq 2$ from a solution.)
*2 pts.* for assuming $\theta = 2$ with no reasons, if (g) solved.

(9) (5+ points) For the labeled Petersen graph $P$ (see my drawing):

(a) (5 points) Find an automorphism (an isomorphism with itself) that carries vertex $a_1$ to $b_2$.

**Solution.** This is a function $f : V(P) \to V(P)$. One solution is

| $v =$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $b_1$ | $b_2$ | $b_3$ | $b_4$ | $b_5$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $f(v) =$ | $b_2$ | $b_4$ | $b_1$ | $b_3$ | $b_5$ | $a_2$ | $a_4$ | $a_1$ | $a_3$ | $a_5$ |

You can describe $f$ with a diagram showing the result of the automorphism.

**Grading rubric.**

*4 pts.* for $f(a_1) = b_1$. (Too easy.)
$-1$ *pts.* for $f^{-1}$.

(b) (Bonus 5 points) How many such automorphisms are there?

**Solution.** The three neighbors of $a_1$ must be mapped to the neighbors of $b_2$. The way you do this is arbitrary, so there are $3! = 6$ ways to do it. Once you've chosen these values of $f$, you have 2 choices for $f(b_3)$, namely the two neighbors of $f(b_1)$. All the rest is determined, as you can see by looking at the graph. So there are $6 \cdot 2 = 12$ automorphisms.

**Grading rubric.**

*2 pts.* for 6 automorphisms. (This is the easy part.)
*0 pts.* for appealing to rotations and inversions. The diagram is not the graph.

(10) (10 points) Use Kruskal's algorithm to find a minimum-weight spanning tree in the graph $G_{10}$.

**Solution.** Kruskal's algorithm picks the edges in the following order:

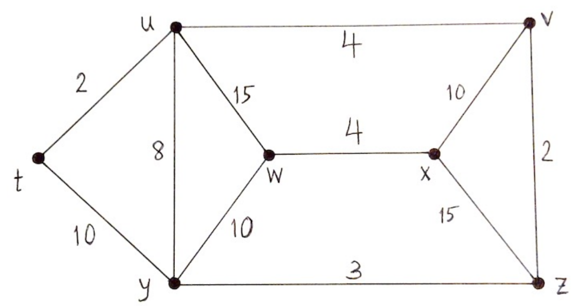
(1) $ut$,
(2) then $vz$ (or the reverse),
(3) then $yz$,
(4) then $wx$,
(5) then $uv$ (or the reverse),
(6) then reject $uy$ because it would form a cycle,
(7) then pick $wy$ (or $xv$, depending on the first one you looked at),
(8) then you have a spanning tree so you can stop.

Step (6) is important: it's part of the algorithm. The weight is 25.

**Grading rubric.**

*7 pts.* for using the other algorithm (tree-growth).
*7 pts.* if not mentioning unused edges that are considered in the algorithm.
*5 pts.* if not clearly stating the order of considering edges, including unused edges.
*3 pts.* for not presenting the steps.
*0 pts.* for no or non-minimum tree.
$-1$ *pts.* for not following the edge order, if you give one in advance.

$G_{10}$




(11) (42 points) For the graph $G_{11}$, get the best results you can.

(a) (10 points) Prove it is nonplanar. You must use Kuratowski's Theorem.
**Solution.** A clearly labeled drawing of a subdivision of $K_5$ or $K_{3,3}$. (Interesting fact: $G_{11}$ has no subgraph that is a subdivision of $K_5$.)
**Grading rubric.**
*2 pts.* for alleged but incorrect subdivision.
*2 pts.* for claiming a $K_5$ or $K_{3,3}$ subgraph (not subdivided). There is none.

(b) (10 points) What is its crossing number $\text{cr}(G_{11})$?
**Solution.** We know $\text{cr}(G_{11}) > 0$ by (a) and $\text{cr}(G_{11}) \leq 2$ by the drawing. Therefore $\text{cr}(G_{11}) = 1$ or 2. (I believe I can prove it is 2, but it's a bit complicated. I will look at this again later.)
**Grading rubric.**
*6 pts.* for $\text{cr}(G_{11}) > 0$ by (a), and claiming $\text{cr}(G_{11}) = 2$ because of a plausible wrong reason, e.g., you tried several drawings and could not get 1. "That is not logical, Captain."
*3 ($-2$ if omitted) pts.* $\text{cr}(G_{11}) > 0$ by (a).

*3 pts.* $\mathrm{cr}(G_{11}) > 0$ by (a); $> 1$ for a reason that doesn't make sense to me or is plainly wrong.

(c) (10 points) What is its splitting number $\sigma(G_{11})$?
**Solution.** By nonplanarity, $\sigma(G_{11}) \geq 1$. One splitting of $g$ (split off the edge $gj$ to become $g'j$) suffices to make $G_{11}$ planar; so $\sigma(G_{11}) \leq 1$. A different splitting is of $f$ (split off edges $fh$ and $fi$). Conclusion: $\sigma(G_{11}) = 1$.
**Grading rubric.**
*3 (−2 if omitted) pts.* $\sigma(G_{11}) > 0$ by (a).
*7 pts.* for $\sigma(G_{11}) > 0$ by (a), and $= 2$ by two splittings.

(d) (10 points) What is its thickness $\theta(G_{11})$?
**Solution.** $\theta(G_{11}) > 1$ by (a). It's easy to decompose $G_{11}$ into two planar subgraphs, so $\theta(G_{11}) = 2$.
**Grading rubric.**
*3 (−2 if omitted) pts.* $\theta(G_{11}) > 1$ by (a).
$-1$ *pts.* per decomposition error.

(e) (2 points) Is it an Earth–Moon graph?
**Solution.** Yes, because the thickness is 2. Draw one planar subgraph on each "planet".
**Grading rubric.**
*0 pts.* for taking "countries" of the nonplanar graph.
*1 pts.* if no reason.

$G_{11}$