Midterm Exam—Math 381—Spring 2021
April 12–13, 2021

GRADING GUIDE

- All answers must be fully justified.
- There are 145 midterm points. (They will be scaled to 100 course points.)

(1) (7 points) If a tree has degree sequence $(8, 8, 5, 5, 4, 2, 1, \dots, 1)$, how many vertices have degree 1?

Solution. *Hint: Name your unknowns!* Let $x =$ number of vertices with degree 1. Then the sum of degrees equals $32 + x = 2q$. The number of vertices equals $6 + x = p = q + 1$. Now, $2q = 32 + x$ and $q = 5 + x$. The solution is $x = 22$.

Grading rubric.

7 pts. General solution, such as above (or with trivial error).
6 pts. General solution, good method, but wrong value of x.
2 pts. Finding one or two trees and getting $x = 22$ from them.
3 pts. A little better than finding a tree and getting $x = 22$ from it.

(2) (5+5 points) A graph (which may not be regular) decomposes into two spanning subgraphs.

(a) (5 points) Suppose both spanning subgraphs are trees. What is the smallest possible degree of a vertex in such a graph? What is the largest possible degree of a vertex? Justify your answers, of course.

Solution. The smallest possible degree is 2, because every vertex is incident with at least one edge from each tree.

The largest possible degree is unlimited. One of the trees can have a vertex of arbitrarily large degree.

Grading rubric.

3 pts. for smallest = 2, with explanation.
2 pts. for largest = unlimited, or ∞, or $n - 1$, with explanation.
1 pts. each without explanation.
+1 pts. bonus for K_1 exception to minimum (though technically it is only one tree).

(b) (5 points) Suppose both spanning subgraphs are paths. What is the smallest possible degree of a vertex in such a graph? What is the largest possible degree of a vertex?

Solution. The smallest: the same reasons and answers as in part (a).

The largest is 4 because the two paths can contribute at most 2 edges each to any vertex.

Grading rubric.

2 pts. for smallest = 2 with explanation.
3 pts. for largest = 4 with explanation.
1 pts. each without explanation.

(3) (10 points) What is the most edges a graph can have if it has p vertices and has a bridge?

Solution. Let G be our graph and let e be the bridge. Then $G-e$ is disconnected, so it has vertex sets A and $B = V - A$ that have no edges between them. The most edges $G-e$ can have is if A and B have all edges, thus $\binom{|A|}{2}$ edges in A and $\binom{|B|}{2}$ edges in B. Let $a = |A|$ so $p - a = |B|$. Thus, $G - e$ has at most $f(a) = \binom{a}{2} + \binom{p-a}{2}$ edges, which simplifies to $f(a) = a^2 - pa + \frac{p^2-p}{2}$. (This is the parabola $y = x^2 - px + \frac{p^2-p}{2}$.) Here a is some integer in the interval $[1, p-1]$. The minimum of f is attained at $a = p/2$, which is the midpoint of the interval, so the maximum is attained at both endpoints (by the symmetry of the parabola), for instance, at $a = 1$, giving $f(1) = 1 - p + \frac{p^2-p}{2} = \binom{p-1}{2}$. (That is the right number by graph theory: if $a = 1$ there are no edges in A and if we have the most edges in B then we have K_{p-1} in the vertex set B.) Thus, G has at most $\binom{p-1}{2} + 1 = \frac{1}{2}(p^2 - 3p + 4)$ edges.

Grading rubric.

10 pts. for this proof.

8 pts. for proof without explaining $\binom{k}{2} + \binom{p-k}{2} \le \binom{p}{2}$.

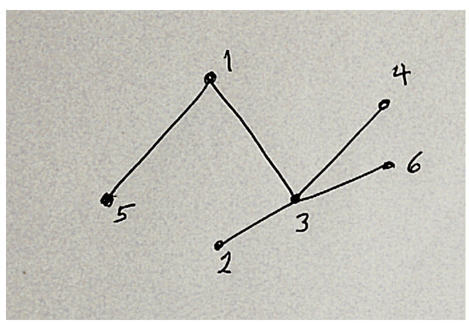
7 pts. for no proof but showing reasoning about K_a and K_{p-a}.

4 pts. for assuming (possibly, implicitly) G must consist of K_{p-1} with one additional edge (the bridge). (We discussed that in class!)

4 pts. for assuming G must consist of two $K_{p/2}$'s with one additional edge (the bridge).

0 pts. for computing the number of edges in K_p. That is not what this question is about.

(4) (7 points) How many 3's are in the Prüfer code for this tree? How does that compare to the degree of vertex 3?



Solution. First find the code: $(3, 3, 1, 3)$ by deleting vertices 2, 4, 5, 1 in that order. The number of 3's is 3 (a coincidence).

Then compare: it equals $\deg(v_3) + 1$.

Grading rubric.

I didn't expect complete work if you showed the right code (but normally work is required).

6+1 pts. for the code (6) and comparison (1) (not just stating the numbers).

0+1 pts. Wrong or no code, no work, no partial credit.

+1 pts. bonus, for mentioning the pattern for all vertices.

(5) (5 points) What is the degree of vertex 2 in the labelled tree whose Prüfer code is $(3, 2, 3, 7, 4)$?

Solution. First find the tree. The code has length 5 so there are 7 vertices: $V = \{1, 2, 3, 4, 5, 6, 7\}$.

Step 1. Add edge 13, leaving code $(2, 3, 7, 4)$ and $V = \{2, 3, 4, 5, 6, 7\}$.
Step 2. Add edge 52, leaving code $(3, 7, 4)$ and $V = \{2, 3, 4, 6, 7\}$.
Step 3. Add edge 32, leaving code $(7, 4)$ and $V = \{3, 4, 6, 7\}$.
Step 4. Add edge 73, leaving code (4) and $V = \{4, 6, 7\}$.
Step 5. Add edge 46, leaving code () and $V = \{4, 7\}$.
Step 6. Add edge 47.
Then the degree of 2 is 2 (coincidence).

Grading rubric.

5 pts. Find tree, answer question.
4 pts. Find tree, answer question. Tree is not quite correct.
5 pts. Answer question by citing the fact from the book. (Maybe not as good, because the book doesn't give a proof.)
0 pts. No work.

(6) (10 points) Prove that every complete graph K_n $(n \geq 2)$ is critical.

Solution. First, $\chi(K_n) = n$ because every vertex has to have a different color (because all vertices are adjacent to each other).

Second, we prove every proper subgraph has smaller chromatic number. It is sufficient to prove $K_n - uv$ can be colored in $n - 1$ colors, for any edge uv. To color $K_n - uv$ with $n - 1$ colors, give every vertex a different color except that u and v get the same color.

That proves K_n is critical.

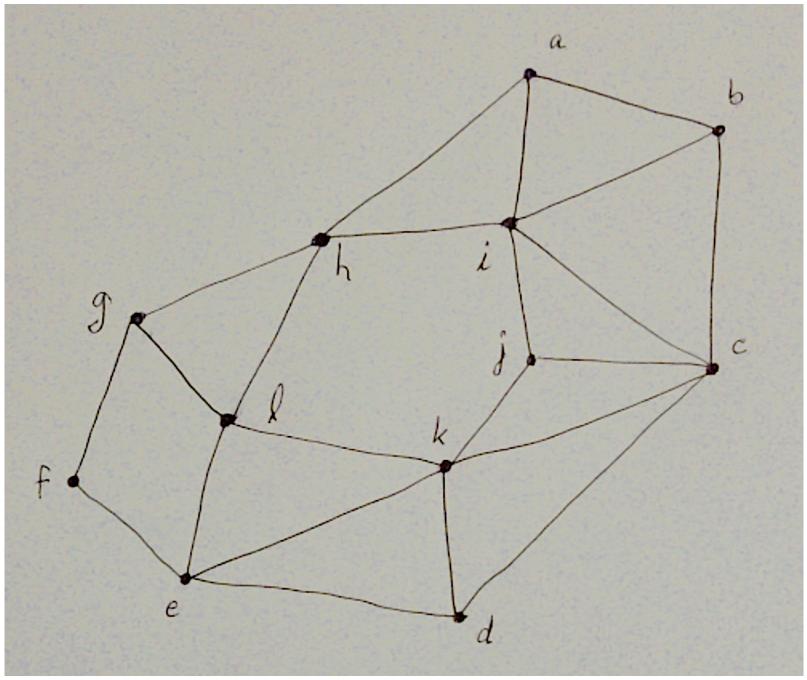
Note that a good proof gives the exact reason that $K_n - uv$ has chromatic number $< n$, which depends on having the exact reason the chromatic number of K_n is n. It is not enough just to cite the fact that $\chi(K_n) = n$.

Grading rubric.

2+8 pts. A full proof requires proving $\chi(K_n)$. The largest proper subgraphs of a graph G are $G - e$ for each edge e (if G has no isolated vertices), so check $K_n - uv$.
2+3 pts. $K_n - v$ instead of $K_n - uv$.
4 pts. for knowing to look at $K_n - uv$, but no proof.

(7) (10 points) Find the chromatic number $\chi(G_7)$ of this graph.

G_7



Solution. The triangles imply the chromatic number is at least 3, so I try to color $\chi(G_7)$ with 3 colors. I can start with any triangle, say $\triangle ahi$ with colors $1, 2, 3$ on a, h, i, respectively. Then $\triangle aib$ implies color 2 on b, $\triangle ibc$ implies color 1 on c, $\triangle icj$ implies color 2 on j, $\triangle abi$ implies color 3 on k, $\triangle abi$ implies color 2 on d, $\triangle abi$ implies color 1 on e, $\triangle abi$ implies color 2 on l. But that gives edge hl the same color at both ends, so 3 colors are not enough.

A procedure like this is needed in order to prove 3 colors are not sufficient.

Now I can see how to color with 4 colors. Let l have color 4, f have color 2, and g have color 3. Thus, G_7 is 4-colorable, which implies $\chi(G_7) = 4$.

Note: Trying to use the least number of colors at each step is not a valid procedure for getting the chromatic number. It can give too big an answer, although in this graph it does not.

Grading rubric.

2 pts. for observing why $\chi(G_7) \geq 3$.
+5 pts. for proving 3 colors are not enough. (+3 for incomplete proof.)
+3 pts. for proving 4 colors are enough.
-3 *pts.* for using the fewest colors at each step instead of trying for 3 colors (so, no credit for noting $\chi(G_7) \geq 3$).
3 pts. for only coloring in 4 colors.
2+3+3 pts. for 3-coloring where an edge is missing.

(8) (10 points) Find the chromatic polynomial of $K_4 - e$, where e is an edge.

Solution. First method: Let $V = \{v_1, v_2, v_3, v_4\}$ and $e = v_3v_4$. Assume we have k colors available. Color v_1 first: there are k colors to choose from. Now color v_2: because it is adjacent to v_1 there are $k-1$ colors to choose from; so there are $k(k-1)$ ways to color the first two vertices. Now color v_3: since it is adjacent to v_1 and v_2, which are different colors, there are only $k-2$ colors to choose from; so there are $k(k-1)(k-2)$ ways to color these three vertices. Finally, for v_4 there are also $k-2$ colors available, since it is not adjacent to v_3; so there are $k(k-1)(k-2)^2$ ways to color the four vertices. In other words, the chromatic polynomial is

$$k(k-1)(k-2)^2 = k^4 - 5k^3 + 8k^2 - 4k.$$

Solution. Second method: As explained on the Chromatic Polynomials handout, the chromatic polynomial $\chi_{K_n-e}(k) = \sum_{m=1}^{4} \psi_m(K_4 - e)\binom{k}{m}$. Since the triangles imply $\chi(K_n - e) \geq 3$, $\psi_1 = \psi_2 = 0$. ψ_3 is the number of ways to use exactly 3 colors, where v_3 and v_4 must have the same color; thus $\psi_3 = 3!$, the number of ways to assign 3 colors to the vertices. $\psi_4 = 4!$, the number of ways to assign 4 colors to the vertices. Therefore, $\chi_{K_n-e}(k) = 3!\binom{k}{3} + 4!\binom{k}{4} = k(k-1)(k-2) + k(k-1)(k-2)(k-3) = k(k-1)(k-2)^2$.

Grading rubric.

4 pts. for some good reasoning but serious error.
8 pts. if mostly good but some gap.
0 pts. for the polynomial of K_4.
-1 pts. for not giving a single polynomial.

(9) (10 points) A graph G is 8-regular and has no 1-factor. Prove that its edge chromatic number is 9.

Solution. By Vizing's theorem, since the maximum degree $\Delta = 8$, the edge chromatic number $\chi' = 8$ or 9. Suppose the edges can be colored with 8 colors. Then every color appears on an edge at every vertex, so the set of edges of each color is a 1-factor. But there is no 1-factor, so the edges can't be colored in 8 colors. So, $\chi' = 9$.

Solution. Alternatively, there are $8p/2 = 4p$ edges. If the edges are colored with 8 colors, the average number of edges per color is $4p/8 = \frac{1}{2}p$. The edges of each color can't share vertices, so the average number of vertices per color is p. But that number can't be more than p for any color, so every color uses p vertices and therefore exactly $\frac{1}{2}p$ edges. Thus, the edges of each color use all vertices, and form a 1-factor. But there is no 1-factor, so coloring in 8 colors is impossible. So, $\chi' = 9$.

Note: There is no reason to think G has odd order.

Grading rubric.

8 pts. for not explaining why every color has $p/2$ edges (if there is an 8-edge-coloring).
2 pts. for assuming $G = K_9$ and using Theorem 2.2.4 or a direct proof.
0 pts. for only citing Vizing's theorem.
0 pts. for vertex coloring.

(10) a. (5 points) What is the most edges a graph that has 10 vertices and no triangles can have?

Solution. By Turan's theorem, $K_{5,5}$ has the most edges. The number of edges is $5 \times 5 = 25$.

Grading rubric.

4 pts. for not stating the number.
2 pts. for only comparing different $K_{r,s}$ with $r + s = 10$.
3 pts. for only girth 4.
1 pts. for ad hoc reasoning.

b. (5 points) What is the most edges in a graph that has 10 vertices, no triangles, and no C_4 subgraph?

Solution. This graph has girth at least 5. The Petersen graph P has girth 5, 10 vertices, and 15 edges. It is the obvious starting point (if you think of it).
The question remains, whether a graph with girth ≥ 5 can have more edges. The argument in the book that shows P is the unique 5-cage shows P is the unique such cubic graph and it can be adapted to prove that more edges are not possible even if you allow higher degrees. Any such proof gets bonus credit.

Grading rubric.

5+10 pts. (bonus) for a complete proof that P is the unique such graph with the most edges.
5+5 pts. (bonus) for a proof that P has the most edges.
5+3 pts. (bonus) for a proof (citing the book) that P uniquely has the most edges for a cubic graph.
5 pts. for P, without proof.
2 pts. for a tree with 9 edges.
3 pts. for a graph with 10 to 14 edges.
0 pts. for a graph that doesn't meet the requirements, or for no graph (I can't check it).

(11) (10 points) Here is a graph theory problem:

(T) We have a graph G. Color its vertices so vertices at distance ≤ 2 have different colors.

a. (5 points) How should you modify G to a graph G' so that problem (T) becomes ordinary coloring of G'?

Solution. Add an edge between each pair of vertices at distance 2, giving G'.

Grading rubric.

5 pts. for a clear statement equivalent to this.
2 pts. for an example.
0 pts. for trying to modify G to a graph G' which needs $\chi(G)$ colors with the new (distance-2) coloring rule.

b. (5 points) When you apply your modification to the Petersen graph (Figure 4.2.3), what graph do you get? Describe it as simply as you can.

Solution. K_{10}. (Reason: you can see it. Or, because the distance between vertices in the Petersen graph is at most 2.)

Grading rubric.
5 pts. for K_{10}, no explanation needed.
4 pts. for a 9-regular graph or other indirect description of K_{10}.

(12) (20+ points) Do *either* (A) or (B). I will count only one of (A) and (B). If you do both, I will count the better one. (B) is harder. Use your time wisely.

(A) (20 points) This is about the complete bipartite graph $K_{r,s}$ (with $r, s > 0$). Let R be the first vertex set, with r vertices; let S be the set of s vertices. (Hint: draw an example to help, such as $K_{3,5}$.)
Give all answers in terms of r, s, in the simplest form you can find.

a. (2 points) What are the degrees of the vertices in $K_{r,s}$?
Solution. For $v \in R$, $\deg v = s$. For $w \in S$, $\deg w = r$.
Grading rubric.
0 pts. if backwards.
1 pts. for no indication of which is which.
2 pts. for only: r vertices of degree s and s vertices of degree r.

b. (3 points) State the conditions under which $K_{r,s}$ has a decomposition into cycles.
Solution. Both r and s are even.
Grading rubric.
0 pts. for $r = s = 2$.
-2 *pts.* for false reason.

c. (3 points) State the conditions under which $K_{r,s}$ has an Eulerian circuit.
Solution. Both r and s are even.
Grading rubric.
0 pts. for $r = s = 2$ or $r = s =$ even number.

d. (6 points) State the conditions under which $K_{r,s}$ has an Eulerian trail.
Solution. $r = 2$ and s is odd, or the reverse, or $r = s = 1$.
I allow you to omit the reverse, because $K_{r,s} = K_{s,r}$.
I accept: Also if both are even; however, an Eulerian circuit is not really an Eulerian trail.
Congratulations if you caught the case of $K_{1,1}$.
Grading rubric.
5 pts. if omit $r = s = 1$.
2 pts. for only $r = s = 1$ or (some special cases).

e. (6 points) What is the minimum number of trails into which $K_{r,s}$ can be decomposed?
Solution. There are three cases. (They can be stated as more cases. The important thing is to cover all cases.)
Case I. 1 trail if r and s are even, by the Euler–Hierholzer theorem.
Case II. $\frac{1}{2}s$ if r is odd and s is even, or $\frac{1}{2}r$ if r is even and s is odd, by Listing's theorem.

Case III. $\frac{1}{2}(r+s)$ if both r and s are odd, by Listing's theorem.

Grading rubric.

-1 *pts.* for using floor or ceiling function. Those integers must be even.
-1 *pts.* for omitting the even-even case.
3 pts. for omitting r, s odd.
0 pts. for only treating one trail, or only examples.

(B) (32=20+12 bonus points) This is about the complete tripartite graph $K_{r,s,t}$ (with $r, s, t > 0$). Let R be the first vertex set, with r vertices; let S be the set of s vertices; and let T be the set with t vertices. (Hint: draw an example to help, such as $K_{3,5,2}$.)
Give all answers in terms of r, s, t, in the simplest form you can find.

a. (2 points) What are the degrees of the vertices in $K_{r,s,t}$?
Solution. $d(x) = s+t$ for $x \in R$; also $d(y) = r+t$ for $y \in S$; and $d(z) = r+s$ for $z \in T$.
Grading rubric.
0 pts. if mixed up.
1 pts. for no indication of which is which.
2 pts. if you don't say how many vertices have each degree.

b. (5 points) State the conditions under which $K_{r,s,t}$ has a decomposition into cycles.
Solution. All of $r+s$, $r+t$, and $s+t$ must be even. That means all of r, s, t have the same parity (even or odd).
Grading rubric.
3 pts. if you stop at $r+s$ etc.
2 pts. for r, s, t must all be even.
-4 *pts.* for false reason.

c. (5 points) State the conditions under which $K_{r,s,t}$ has an Eulerian circuit.
Solution. Identical to (b).
Grading rubric.
Identical to (b).

d. (10 points) State the conditions under which $K_{r,s,t}$ has an Eulerian trail.
Solution. To have an Eulerian trail (not an Eulerian circuit), not all of $r+s$, $r+t$, and $s+t$ can be even. That means not all of r, s, t can have the same parity. There are two cases:

Case I. Suppose one is odd is odd, say r, and the others are even. Then $r+s$ (t vertices) and $r+t$ (s vertices) are odd and $s+t$ (t vertices) is even. Since there must be exactly two odd-degree vertices, $s+t$ must equal 2, so $s = t = 1$. But that contradicts the assumption that only r is odd, so this case is impossible.

Case II. Suppose two are odd, say s, t are odd and r is even. Then $r+s$ (t vertices) and $r+t$ (s vertices) are odd and $s+t$ (r vertices) is even. Then $s+t$,the number of odd vertices, must equal 2, so $s = t = 1$.

Also, r is even. We can state this solution in general as that two of r, s, t equal 1 and the third of them is even.

I allow adding the case that all are odd or all even; however, an Eulerian circuit is not really an Eulerian trail.

Grading rubric.

5 pts. if you don't deduce the solution, such as above (that is the proof), but only check that it works.

7–8 pts. for solution with partial proof.

4 pts. if you only state the solution without proof or check.

e. (10 points) What is the minimum number of trails into which $K_{r,s,t}$ can be decomposed?

Solution. I'll break this into cases.

Case I. r, s, t all have the same parity. There is an Eulerian circuit and the number of trails is 1.

Case II. One of r, s, t is odd and the others are even. Suppose r is the odd one, so s, t are even. Then the odd degrees are $r + s$ (t of them) and $r + t$ (s of them), giving a total of $s + t$ odd-degree vertices. The number of trails is $\frac{1}{2}(s + t)$. We can say the general solution as: Half the sum of the two even numbers among r, s, t.

Case III. Two of r, s, t are odd and the third is even. Suppose s, t are odd and r is even. Then the odd degrees are $r + s$ (t of them) and $r + t$ (s of them), giving a total of $s + t$ odd-degree vertices. The number of trails is $\frac{1}{2}(s + t)$. We can say the general solution as: Half the sum of the two odd numbers among r, s, t.

I could combine Cases II and III by saying: If r, s, t don't all have the same parity, the number of trails is half the sum of the two that do have the same parity.

Grading rubric.

-2 pts. for omitting Case I or saying 0 or > 1 trails.

4 pts. for a limited partial analysis, partial proof, and partial solution.

0 pts. for only treating one trail, or only examples.

(13) (10 points) Find the line graph of $K_{3,3}$.

Solution. Let's say $K_{3,3}$ has left vertices u_1, u_2, u_3 and right vertices v_1, v_2, v_3. The vertices of $L(K_{3,3})$ are the edges $u_i v_j$ for $i, j = 1, 2, 3$. I will describe a picture of $L(K_{3,3})$: Plot the vertex $u_i v_j$ as a point $(x, y) = (i, j)$, forming a 3×3 square of points in the xy-plane. Draw three vertical edges in each column $x = 1, 2, 3$, and three horizontal edges in each row $y = 1, 2, 3$. That is $L(K_{3,3})$.

Grading rubric.

7 pts. for one significant error.

9 pts. if missing one or two edges.

(14) (6 points) We know that if a graph G has an Eulerian circuit, then its line graph $L(G)$ has a Hamilton cycle. Give three examples of a graph that has no Eulerian circuit, but whose line graph has a Hamilton cycle.

Solution. Observations: $L(K_{1,n}) \cong K_n$. No $K_{1,n}$ has an Eulerian circuit. Every K_n with $n \geq 3$ has a Hamilton cycle. I conclude that an infinite family of graphs G is all the star graphs $K_{1,n}$ for $n \geq 3$. You could mention three or more of them.

These are the simplest. There are many other (and more interesting) examples. It was fun to see some of your examples.

Grading rubric.

2 pts. per example. No proof needed (to save time grading), but it has to be correct!

(15) (10 points) A graph G that is regular of degree $r > 0$ decomposes into two (2) Hamilton cycles. What graph is G?

Solution. First, G is 4-regular (so $p \geq 5$). Then, we can't identify G uniquely. We can construct all such graphs by taking any cycle C_p with $p \geq 5$, and let's say the vertices are $v_1, v_2, \ldots, v_p$ in cycle order, then adding a single cycle that goes through all the vertices but never takes two vertices $v_i, vi + 1$ or v_p, v_1 consecutively. This leaves a lot of room for G.

One can say we take any two Hamilton cycles in a decomposition of K_p ($p \geq 5$) into Hamilton cycles (and a 1-factor if p is even), but I don't think this gives every G. If it does, a proof is needed.

I don't remember what I had in mind for this problem. I suspect I intended one additional condition to narrow down the graph, but it vanished somehow.

Grading rubric.

3 pts. for K_5 or any other one graph.

4 pts. for noticing 4-regularity and claiming every 4-regular graph is an example, which is probably not true (it would be a theorem, if true).[1]

4 pts. for noticing 4-regularity and presenting one example.

6 pts. for noticing 4-regularity and more than one example.

8 pts. for two Hamilton cycles from a decomposition of K_p, as above.

[1] It is not true; here is a counterexample graph G. Take two connected, 4-regular graphs, G_1 and G_2, and one edge in each, u_1v_1 in G_1 and u_2v_2 in G_2. Replace those edges by new edges u_1u_2 and v_1v_2; the new graph is G. G is connected and 4-regular. A Hamilton cycle in G must contain both u_1u_2 and v_1v_2. Thus, there cannot be two edge-disjoint Hamilton cycles in G.