The Matrix-Tree Theorem
Math 381
Version of April 5, 2021

Chapter 5 gives formulas for the number of spanning trees in some graphs. There is a remarkable formula for the number of spanning trees in any graph. It involves a matrix associated with a graph. (The matrices here are more important for Math 381 than the theorem.)

Definition MT.1. The *adjacency matrix* of a graph G with vertex set $V = \{v_1, v_2, \ldots, v_p\}$ is the $p \times p$ matrix $A(G) = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0 & \text{if } v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

In particular, the diagonal is all 0, since v_i is not adjacent to itself in a graph.

The adjacency matrix is a symmetric matrix; therefore, by the diagonalization theorem of symmetric matrices, it has all real eigenvalues. We won't go into eigenvalues of adjacency matrices, but there is a lot of research on them right up to now.

Definition MT.2. The *degree matrix* of G is the $p \times p$ matrix $D(G)$ with the degree of v_i on the diagonal in row and column i, and with 0's off the diagonal.

The next matrix is the one we are really interested in, as concerns spanning trees.

Definition MT.3. The *Kirchhoff matrix* of G is the matrix $K(G) = D(G) - A(G)$.

Example MT.1. Let $G = K_4 \setminus v_2v_4$. Then

$$A(G) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad D(G) = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad K(G) = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}.$$

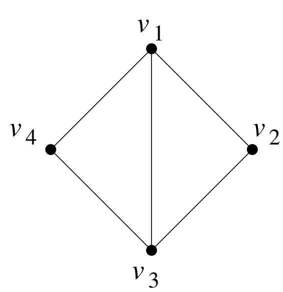

Figure MT.1. $K_4 \setminus v_2v_4$.

Theorem MT.1. *The determinant of the Kirchhoff matrix is zero:* $\det K(G) = 0$.

Proof. The sum of the entries in row i of the adjacency matrix $A(G)$ is $\deg v_i$, by the definition of degree. Therefore, in $K(G)$, the sum of the entries in row i is $\deg v_i - \deg v_i = 0$, for every row. That means the columns of $K(G)$ are linearly dependent, so the determinant is 0 by basic linear algebra. □

This is not why the Kirchhoff matrix is interesting. Let's delete one row and one column from it and *then* take the determinant. We call this matrix K_{ij}, if row i and column j are deleted. For example:

Example MT.2. Apply that procedure to Example MT.1. Let's delete row 1 and column 1.

$$K(G)_{11} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad \det K(G)_{11} = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{vmatrix} = (12+0+0)-(2+2+0) = 8.$$

Another way to do it:

$$K(G)_{13} = \begin{pmatrix} -1 & 2 & 0 \\ -1 & -1 & -1 \\ -1 & 0 & 2 \end{pmatrix}, \quad \det K(G)_{13} = \begin{vmatrix} -1 & 2 & 0 \\ -1 & -1 & -1 \\ -1 & 0 & 2 \end{vmatrix} = (2+2+0)-(0-4+0) = 8.$$

Now count the spanning trees in G. Do you get 8?

Theorem MT.2 (Kirchhoff's Matrix-Tree Theorem)**.** *For any $i, j \in \{1, 2, \ldots, p\}$, the value of $(-1)^{i+j} \det K(G)_{ij}$ is the number of spanning trees in G.*

I will not prove this theorem. In case you want to know, it involves factoring the Kirchhoff matrix into a product $H(G)H(G)^T$ (where $H(G)$ is a matrix called the *incidence matrix* of G), a beautiful matrix theorem called the Cauchy–Binet Theorem that gives a formula for the determinant of a product of the form HH^T (where H is any matrix), and some clever analysis of graph matrices.

Exercise MT.1. (a) Find the adjacency and Kirchhoff matrices of K_3.

(b) Try several combinations of row and column deletions and find their determinants (with the sign factor), i.e., $(-1)^{i+j} \det K(K_3)_{ij}$ for several combinations of i and j. Try at least one case where $i = j$, and at least one where $i \neq j$. You notice that you always get the same value of the (signed) determinant.

(c) Count the spanning trees of K_3 directly in the graph. Compare with (b). They ought to be equal; if not, did you make a mistake?

Exercise MT.2. (a) Find the adjacency and Kirchhoff matrices of C_4, the cycle of length 4.

(b) Try several combinations of row and column deletions and find their determinants, i.e., $(-1)^{i+j} \det K(C_4)_{ij}$ for several combinations of i and j. Try at least one case where $i = j$, and at least one where $i \neq j$. You notice that you always get the same value of the (signed) determinant.

(c) Count the spanning trees of C_4 directly in the graph. Compare with (b). They ought to be equal; if not, did you make a mistake? The most likely mistake is to evaluate a 4×4 determinant incorrectly. Make sure you know how to do it.

Exercise MT.3. Do the same for K_4.

(a) Find $A(K_4)$, $D(K_4)$, and $K(K_4)$.

(b) Write out $K(K_4)_{ii}$ for one choice of i, and evaluate its determinant by using row and column operations to simplify the matrix.

(c) Does your determinant agree with Cayley's formula $s(K_4) = 16$? If not, I predict you made a mistake.

Exercise MT.4. Do the same for K_n, in the following way:

(a) Find $A(K_n)$, $D(K_n)$, and $K(K_n)$.

(b) Write out $K(K_n)_{ii}$ for one choice of i, and evaluate its determinant by using row and column operations to simplify the matrix.

(c) Does your determinant agree with Cayley's formula $s(K_n) = n^{n-2}$ (Theorem 5.2.1)?

If you made no mistake, you have a proof of Cayley's formula by matrix theory.

Exercise MT.5. Do parts (a, b) from Exercise MT.4 for the complete bipartite graph $K_{m,n}$.

Does your result agree with Theorems 5.3.1 ($m = 2$) and 5.3.2 ($m = 3$)?

Exercise MT.6. Do parts (a, b) from Exercise MT.4 for the cycle C_n ($n \geq 3$).

(c) Calculate the number of spanning trees in C_n from the graph itself, and compare with the number you got in (b). They should agree.