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Complementing the theory of rook placements in Section 6.4 of Brualdi's book, there is the...

Theory of Crook Placements

Crooks have nasty habits. They shoot each other on sight (with lasers). Fortunately, they only shoot each other, and they can only see horizontally or vertically.[1]

However, they can see in mirrors. Here is the set-up: a "building" in the shape of a triangular set of squares, called "rooms", like the upper left part of an $n \times n$ matrix (and numbered the same way). The rooms are numbered (i, j) with $1 \le i, j$ and $i + j \le n + 1$—that's what makes it a triangle. Just off the lower right edge is a gigantic mirror, occupying what would be rooms $(1, n+1), (2, n), \ldots, (n+1, 1)$, i.e., rooms $(i, n+2-i)$, if there were rooms there. Looking in the mirror, a crook in room (i, j) can see right up column $n+2-i$ and along row $n+2-j$. (You should check all these numbers in a small example. You don't have to take my word for it!)

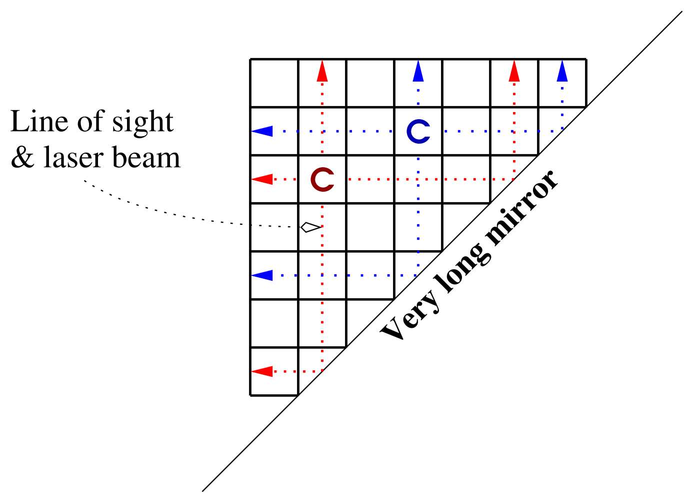

Figure 1. The Realm of Crooks, with two non-attacking crooks: one red, one blue.

What Do We Want to Know?

The big questions are:

Question 1. What is the largest number of crooks that can be on the board without shooting (or turning green)?

Question 2. How many different ways are there to place those crooks? Or, to place k crooks for any $k > 0$?

Question 3. What happens to those questions when you start having rooms with no floors? (No one, not even a crook, can stand on a no-floor.)

[1]If you don't care for shooting, let's say they turn green with envy if they see another crook. A crook always thinks the other crook is stealing better stuff.

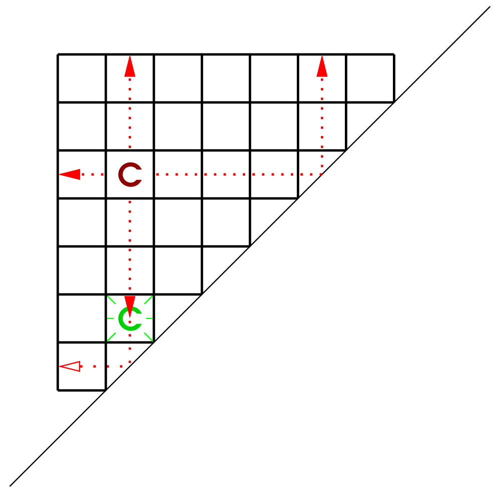

FIGURE 2. Crook turns green with envy when hit by laser beam—or, when see a rival. This is a no-no.

Question 4. What is the relationship between the number of ways k crooks can fit in a building with forbidden rooms (those with no floors), and the number of ways they could fit in if the forbidden rooms became the allowed rooms (the complementary building)?

Let's contrast this with the world of rooks. Rooks live on a nice, rectangular board and they move in nice, straight lines, horizontally and vertically only. No crooked lines for them! It's easy to analyze rooks. On an $n \times n$ board (I'm sticking to square boards, to keep it simple), you can have at most n non-attacking rooks (Question 1), and you can, easily, have n if there aren't too many bad rooms. For crooks, though, it's not so simple.

Let's Have Some Answers.

Here's how to answer Question 1: Use the mirror! This is shown in Figure 3. The board becomes a square board with an extra row and column, so it now has $n+1$ of each. Each crook becomes two: the real crook and the mirror image. They have to be in different rows (and different columns) because the crook in row i, column j has his reflection in row $n+2-j$, column $n+2-i$. (You can see that by solving the equations of reflecting square (i,j) in the mirror, whose equation is $x+y=n+1$.) Now, since every crook (and reflection) needs 2 rows of the real + reflected board, and there are $n+1$ rows in that board, the most rooks you can have is $\lfloor (n+1)/2 \rfloor$, which is written more simply as $\lceil n/2 \rceil$.

You might (if you're cautious) ask whether it really is possible to have $\lceil n/2 \rceil$ crooks in the building without mayhem. Answer: Yes. (Of course; what else?) Can you find an easy way to put that many crooks in rooms? (I'm assuming there's a floor in every room.)

Here's the answer to Question 4. Write q_k^{crooks} for the number of ways to put k crooks on the board, and r_k^{crooks} for the number of ways to put them on the complementary board. (Note that $q_0^{\text{crooks}} = r_0^{\text{crooks}} = 1$.) Then

$$q_k^{\text{crooks}} = \sum_{i=0}^{k} (-1)^i \binom{\frac{1}{2}(n+1)-i}{k-i} \binom{\frac{1}{2}n-i}{k-i} 2^{k-i}(k-i)!\, r_i^{\text{crooks}}. \tag{1}$$

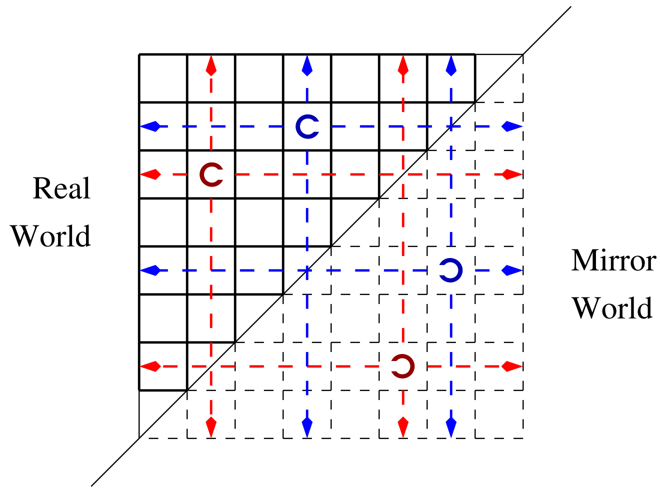

FIGURE 3. The Real and Mirror Worlds of the crooks, with two real and mirror crooks, one red and one blue. Watch those lines of sight!

You can see the fractions in the binomial coefficients. For comparison, in the rooks problem the formula is

$$q_k^{\text{rooks}} = \sum_{i=0}^{k} (-1)^i \binom{n-i}{k-i}^2 (k-i)!\, r_i^{\text{rooks}}. \tag{2}$$

You can answer Question 2 by arranging to have no bad squares. That means $r_0^{\text{rooks}} = 1$ but $r_1^{\text{rooks}} = r_2^{\text{rooks}} = \cdots = 0$. Then from Equation (1) we find out that the number of ways to place k non-attacking crooks with no forbidden squares is

$$q_k^{\text{crooks}}(\text{complete triangular board}) = \binom{\frac{1}{2}(n+1)}{k}\binom{\frac{1}{2}n}{k} 2^k\, k!. \tag{3}$$

For comparison, the number of ways to place k rooks on an $n \times n$ board with no forbidden squares is

$$q_k^{\text{rooks}}(\text{complete square board}) = \binom{n}{k}^2 k!. \tag{4}$$

That leaves Question 3, but I don't know any short answer. Just as with rooks, it all depends on which rooms have no floors, and how they are arranged. Some patterns of bad rooms have nice answers. Some don't. Oh, well.

Few Crooks.

Before I get to a particular example, let's see what Equation (1) looks like for small values of k:

$$\begin{aligned}
q_0^{\text{crooks}} &= (-1)^0 \binom{\frac{1}{2}(n+1)-0}{0}\binom{\frac{1}{2}n-0}{0} 2^0\, 0!\, r_0^{\text{crooks}} \\
&= 1 \quad \text{(which is not a surprise)},
\end{aligned}$$

$$\begin{aligned}
q_1^{\text{crooks}} &= (-1)^0 \binom{\frac{1}{2}(n+1)-0}{1}\binom{\frac{1}{2}n-0}{1} 2^1\, 1!\, r_0^{\text{crooks}} \\
&\qquad\qquad + (-1)^1 \binom{\frac{1}{2}(n+1)-1}{0}\binom{\frac{1}{2}n-1}{0} 2^0\, 0!\, r_1^{\text{crooks}} \\
&= \frac{n+1}{2}\,\frac{n}{2}\,2 - 1\, r_1^{\text{crooks}} \\
&= \binom{n+1}{2} - r_1^{\text{crooks}} \quad \text{(also not surprising)},
\end{aligned}$$

$$\begin{aligned}
q_2^{\text{crooks}} &= (-1)^0 \binom{\frac{1}{2}(n+1)-0}{2}\binom{\frac{1}{2}n-0}{2} 2^2\, 2!\, r_0^{\text{crooks}} \\
&\qquad\qquad + (-1)^1 \binom{\frac{1}{2}(n+1)-1}{1}\binom{\frac{1}{2}n-1}{1} 2^1\, 1!\, r_1^{\text{crooks}} \\
&\qquad\qquad\qquad + (-1)^2 \binom{\frac{1}{2}(n+1)-2}{0}\binom{\frac{1}{2}n-2}{0} 2^0\, 0!\, r_2^{\text{crooks}} \\
&= \binom{\frac{n+1}{2}}{2}\binom{\frac{n}{2}}{2} 8 - \frac{n-1}{2}\,\frac{n-2}{2}\, 2\, r_1^{\text{crooks}} + r_2^{\text{crooks}} \\
&= \left[\frac{n+1}{2}\,\frac{n-1}{2}\right]\left[\frac{n}{2}\,\frac{n-2}{2}\right] 2 - \binom{n-1}{2} r_1^{\text{crooks}} + r_2^{\text{crooks}} \\
&= 3\binom{n+1}{4} - \binom{n-1}{2} r_1^{\text{crooks}} + r_2^{\text{crooks}},
\end{aligned}$$

$$\begin{aligned}
q_3^{\text{crooks}} &= (-1)^0 \binom{\frac{1}{2}(n+1)-0}{3}\binom{\frac{1}{2}n-0}{3} 2^3\, 3!\, r_0^{\text{crooks}} \\
&\qquad\qquad + (-1)^1 \binom{\frac{1}{2}(n+1)-1}{2}\binom{\frac{1}{2}n-1}{2} 2^2\, 2!\, r_1^{\text{crooks}} \\
&\qquad\qquad\qquad + (-1)^2 \binom{\frac{1}{2}(n+1)-2}{1}\binom{\frac{1}{2}n-2}{1} 2^1\, 1!\, r_2^{\text{crooks}} \\
&\qquad\qquad\qquad\qquad + (-1)^3 \binom{\frac{1}{2}(n+1)-3}{0}\binom{\frac{1}{2}n-3}{0} 2^0\, 0!\, r_3^{\text{crooks}} \\
&= 15\binom{n+1}{6} - 3\binom{n-1}{4} r_1^{\text{crooks}} + \binom{n-3}{2} r_2^{\text{crooks}} - r_3^{\text{crooks}}.
\end{aligned}$$

The Example.

Here's a very specific example. The building has side 6 and some bad rooms, as you see in Figure 4. I'd like to get 3 crooks, the maximum, to work stealing jewels in this building! I'll start them each in a room so they can't see each other. How many ways are there I might assign their rooms? I don't want them to see each other, remember! No point in wasting crooks.

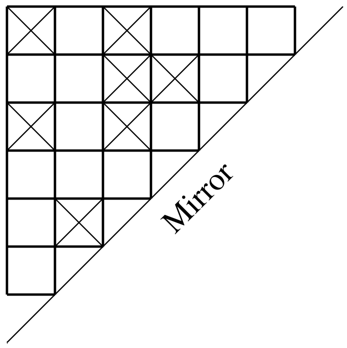

FIGURE 4. Plan of the building in the example.

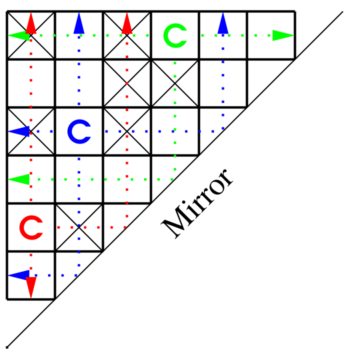

FIGURE 5. One way to set out 3 crooks in the example.

In this example we find (going down columns, from left to right) that

$$r_1^{\text{crooks}} = 7,$$
$$r_2^{\text{crooks}} = 4+4+1+1+0+1+0 = 11,$$
$$r_3^{\text{crooks}} = (1+0+1)+(1+1)+0 = 4.$$

Therefore, by the complementation formula,

$$q_1^{\text{crooks}} = \binom{7}{2} - r_1^{\text{crooks}} = 16,$$
$$q_2^{\text{crooks}} = 3\binom{7}{4} - \binom{5}{2} r_1^{\text{crooks}} + r_2^{\text{crooks}} = 46,$$
$$q_3^{\text{crooks}} = 15\binom{7}{6} - 3\binom{5}{4} r_1^{\text{crooks}} + \binom{3}{2} r_2^{\text{crooks}} - r_3^{\text{crooks}} = 29.$$