HANDOUT I: CONSTRUCTION OF MAGIC SQUARES

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1. Magic squares

Definition 1. A *magic square M of order n* is an $m \times m$-array filled with the numbers $1, 2, \dots, n^2$ so that each number appears exactly once, and with the requirement that the sum of the numbers over any row is equal, and equals the sum over any column and along the two main diagonals. This sum is called the *magic sum* associated to M and is denoted by S_M.

Remark 1. Since the numbers of every row of a magic square M sum up to S_M and the sum over all rows is the sum $1 + 2 + 3 + \ldots + n^2 = \frac{1}{2}n^2(n^2 + 1)$, we know that S_M has to be one n-th of this total. Thus, every magic square M of order m has the same magic sum, namely

$$S_M = \frac{n(n^2 + 1)}{2}.$$

There is exactly one magic square of order 1, and there is no magic square of order two (the magic sum would be 5). We want to address the problem of constructing a magic square of every given order.

Example 1. A magic square of order 3, and one of order 4:

8	1	6
3	5	7
4	9	2

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Exercise 1. Construct another magic square of order 4.

2. "Prime" magic squares

We will indicate in this section a method to construct magic squares of order p, when p is a prime number bigger than 3.

Definition 2. A *quasimagic square* of order n is a square array Q of n^2 boxes filled with natural numbers, such that the sums along every row, every column, and both diagonal are all equal. This quantity is called the *magic sum*, S_Q.

Thus, a magic square is a quasimagic square with the additional requirement that the boxes are filled using exactly once all the numbers from 1 to n^2. The important point is that the *sum* of two quasimagic squares is again a quasimagic square. The sum of two squares is the square obtained by taking in each box the sum of the entries of that box in each of the squares. The sum of the two squares Q_1 and Q_2 is written $Q_1 + Q_2$.

Example 2. Two quasimagic squares and their sum:

$$\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{matrix} \;+\; \begin{matrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{matrix} \;=\; \begin{matrix} 3 & 5 & 6 \\ 4 & 5 & 4 \\ 5 & 3 & 4 \end{matrix}$$

Now let $p > 3$ be a prime number. We are going to build our magic square M of order p as the sum of two quasimagic squares, Q_1 and Q_2. We need to design Q_1 and Q_2 so that their sum contains exactly once the numbers $1, \dots, p^2$. To this end, consider the sets

$$\{1, 2, \dots, p\}, \quad \{0, p, 2p, \dots, (p-1)p\}$$

and notice that by taking *all* possible sums between elements of these sets we get exactly once every number between 1 and p^2.

We thus have to construct a quasimagic square Q_1 with the numbers $1, \dots, p$ and a quasimagic square Q_2 with entries from $0, 1, 2, \dots, (p-1)$ so that *the pairs of numbers associated to the same entries of Q_1 and Q_2 never repeat* – that is, every entry corresponds to a unique pair of numbers, and so to a unique sum. Once we have done that, we will just have to evaluate

$$M = Q_1 + pQ_2.$$

2.1. **Construction of Q_1.** The first row of Q_1 will be given by the numbers $1, 2, \dots, p$ arranged in increasing order. The second row is obtained by shifting the first row (to the left) by some number k of places. The third row is obtained from the second the same way the second was constructed from the first – and so on. For example, if $k = 2$, we obtain

$$Q_1 = \begin{matrix} 1 & 2 & 3 & \cdots & p-2 & p-1 & p \\ 3 & 4 & 5 & \cdots & p & 1 & 2 \\ 5 & 6 & 7 & \cdots & & 3 & 4 \\ \vdots & & & & & & \vdots \end{matrix}$$

Fact 1. Since p is a prime number, for every choice of $0 < k < p$ we will obtain an array with every number from 1 to p appearing exactly once in each row and each column. To ensure that the down-left-to-top-right diagonal also contains every number once, we must require $k > 2$. To ensure the same for the other diagonal, $k < p$ will be enough.

Exercise 2. Give a proof of Fact 1.

Notice that it is here that we need to assume that $p > 3$. For if $p = 3$ there is no number k that can meet the requirements!

Thus, we truly have a quasimagic square with sum

$$S_{Q_1} = \frac{p(p+1)}{2}.$$

2.2. **Construction of Q_2.** We apply to the sequence $0, 1, 2 \dots, p-1$ the same shifting method we used for Q_1, with one caveat: we cannot take the same shift as above, or else we would have plenty of repeated pairs of entries, which, as we pointed out, we do not want to happen.

We will then let

$$Q_2 := \begin{matrix} 0 & 1 & \cdots & p-1 \\ k' & k'+1 & \cdots & k'-1 \\ 2k' & 2k'+1 & \cdots & 2k'-1 \\ \vdots & & \cdots & \vdots \end{matrix}.$$

where k' must satisfy $1 < k' < p-1$ (by the general rule for the shift) and $k' \neq k-1$ (in order not to have the same shift as in Q_1).

This concludes the construction of magic squares of prime order bigger than 3.

3. Product of magic squares

Here we describe a procedure that, given a magic square M of order m and a magic square N of order n, allows us to construct a magic square of order mn. Together with the previous section and the two magic squares of Example 1, we thus have a method that allows us to construct magic squares for every odd order and for every *doubly even* order, i.e., order $4k$ for some $k > 0$.

Definition 3. Let a magic square M of order m and a magic square N of order n be given. Let $M_{i,j}$ be the entry in the i-th row and j-th column of M, and let $N_{i,j}$ be defined analogously.

Given any integer number k, let us define the quasimagic square $M^{(k)}$ as the array obtained by adding $(k-1)m^2$ to every entry of M, so that for all i, j we have

$$M^{(k)}_{i,j} := M_{i,j} + (k-1)m^2.$$

The *composed* magic square M^N is obtained by creating an $n \times n$ grid of $m \times m$ squares, more precisely the square in the i-th row and j-th column of the grid is $M^{(N_{i,j})}$.

Theorem 1. *M^N is a magic square of order nm.*

Proof. The proof consists basically in checking that every row, column and diagonal amounts to the magic sum of squares of order mn, namely $\frac{1}{2}mn(m^2n^2+1)$. We will do this for rows, leaving the check for columns and diagonals to the reader.

So pick a row of M^N, let's say it's the i-th row in the h-th row of "M^ks". Then the sum of the entries is given by the sums of the entries of the i-th rows of all $M^{(N_{h,j})}$, for $j = 1, \dots, n$. Since the $M^{(k)}$s are obtained from a magic square by adding the constant term $(k-1)m^2$, their constant sum over rows $S_{M^(k)}$ equals $S_M + (k-1)m^3$:

$$
\begin{aligned}
\sum_{j=1}^{n} S_{M^{(N_{h,j})}} &= \sum_{j=1}^{n} S_M + (N_{h,j}-1)m^3 \\
&= nS_M + m^3 \underbrace{\sum_{j=1}^{n} N_{h,j}}_{S_N} - nm^3 \\
&= n\frac{m(m^2+1)}{2} + m^3\frac{n(n^2+1)}{2} - nm^3 \\
&= \frac{1}{2}(nm + nm^3 + m^3n + m^3n^3 - 2nm^3) \\
&= \frac{nm(n^2m^2+1)}{2} = S_{M^N}. \qquad \square
\end{aligned}
$$

4. Magic squares of even order

The construction of magic squares of even order turns out to be more difficult. The argument above (subdividing M as a sum $Q_1 + Q_2$) actually works here too, but it needs to be refined with lots of technical adjustments. We will thus not carry out the full argument here – the interested reader will find a thorough explanation in [3].

We will only illustrate two methods, without explaining them.

4.1. Squares of order $4k$, $k > 0$.

The method for this case is illustrated in exercise 1.8.18 of Brualdi's *Introductory Combinatorics* [1], and we will not repeat it here.

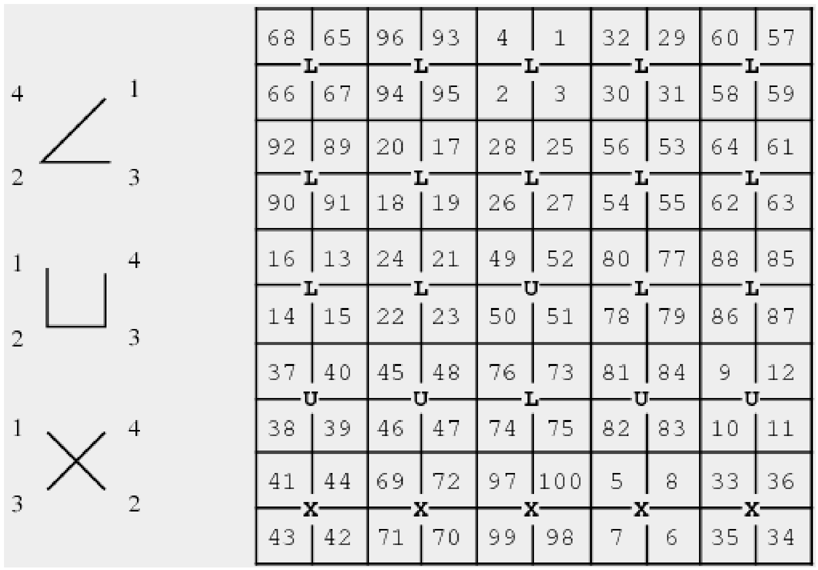

FIGURE 1

4.2. **Squares of order $4k+2$, $k>0$: *"fiat LUX"*.** This method is due to Conway [2]. It starts by subdividing the square into 2×2-subsquares, and putting at the intersection of the lines of each such block the letter L, U or X, according to the following rules:

(1) Put L's in the first $m+1$ rows.
(2) Put U's in the $m+2$-nd row.
(3) Put X's in the remaining rows.
(4) *Switch the middle* U *with the* L *above it.*

Now we use De La Loubère's method to generate a magic square of order $2m+1$. The sequence of the numbers in the boxes of this square will give us a sequence of 2×2 boxes in the big square. We will then fill in the big square with the numbers $1, 2, \ldots, 4m+2$ in increasing order, filling the 2×2 boxes in the order given by De La Loubère, and within each 2×2 filling the squares in the sequence "naturally" suggested by the letters L, U, X – as in Figure 1. That figure shows the patterns for the letters and an example of the construction.

REFERENCES

[1] R. A. Brualdi, *Introductory Combinatorics*, 4th edition. Prentice Hall, Upper Saddle River, N.J., 2004.
[2] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning Ways for Your Mathematical Plays*, Vol. 2: Games in Particular. Academic Press, London, 1982.

[3] A. Delesalle, *Carrés magiques.* Gauthier-Villars, Paris, 1956.
In the Binghamton University library under catalog number QA 165 D347.