Stirling's Approximation

First, Stirling's approximation for $n!$; then the approximation to D_n.

1. Stirling's Approximation to the Factorial

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}. \tag{1}$$

This is the simplest approximation used when you only need a good estimate. The precise meaning of $\approx$ in (1) is that the *quotient* of these two quantities approaches 1:

$$\lim_{n\to\infty} \frac{n!}{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}} = 1.$$

The *difference* is a different story; it gets large. A simple estimate of the difference is:

$$n! - \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \;\approx\; \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\, \frac{1}{12n}. \tag{2}$$

Written as an approximation of $n!$:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \left[1 + \frac{1}{12n}\right]. \tag{3}$$

The exact infinite series is:

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \left[1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{571}{2488320n^4} + \cdots\right]. \tag{4}$$

This series does not converge. You may wonder what good a non-convergent series is. That's a good question. If you take a certain number of terms of the series, you get a very good approximation to $n!$. If you take too many terms, you get a terrible approximation. The best number of terms to take depends on n; the bigger n is, the more terms you need for the best approximation. This gets very complicated, so mostly we just use (1).

2. Approximating the Derangement Number

From (1) and $D_n = [[n!/e]]$ (here $[[\ ]]$ means take the nearest integer) we get

$$D_n \approx \frac{n^n}{e^{n+1}}\sqrt{2\pi n}. \tag{5}$$

The precise meaning of $\approx$ in (5) is that the quotient approaches 1:

$$\lim_{n\to\infty} \frac{D_n}{\frac{n^n}{e^{n+1}}\sqrt{2\pi n}} = 1.$$