Product Behavior of Avoidance Counts
(Chapter 17, Problem 23)
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The problem is to show that $S_n(q)S_m(q) \leq S_{n+m}(q)$, where q is any pattern and $n, m > 0$. The solution is cute.

Define

$$\mathcal{S}_n(q) := \{p \in \mathrm{Perm}_n \mid p \text{ avoids } q\},$$

so $S_n(q) = |\mathcal{S}_n(q)|$. We want to prove that $|\mathcal{S}_n(q) \times \mathcal{S}_m(q)| \leq |\mathcal{S}_{n+m}(q)|$, which we can prove by producing an injective function

$$\theta : \mathcal{S}_n(q) \times \mathcal{S}_m(q) \to \mathcal{S}_{n+m}(q).$$

The hard part is to find the function, that is, to find a definition of $\theta(r, s)$ for $(r, s) \in \mathcal{S}_n(q) \times \mathcal{S}_m(q)$ such that $\theta(r, s)$ avoids q.

There are two cases. Let $l :=$ the length of q, i.e., q is an l-permutation. So, the largest element in q is l, and obviously the smallest is 1. In q, either 1 precedes l, or l precedes 1.

Case 1. 1 precedes l in q. Then define $\theta(r, s) := r \otimes s$, where $r \otimes s$ is the $n+m$-permutation obtained by adding m to every element of r, call this r', and then concatenate r' and s, forming $r's$. Pictorially, that means we put r before s but higher.

Case 2. l precedes 1 in q. Then define $\theta(r, s) := r \oplus s$ as defined in the book (Definition 14.13); that is, $r \oplus s$ is the $n+m$-permutation obtained by adding n to every element of s, call this s', and then concatenate r and s', forming rs'. Pictorially, we put r before s but lower.

Now I have to prove the resulting permutation avoids q.

Proof for Case 1. Here $q_i = 1$ and $q_j = l$ where $i < j$. Suppose we find a q-pattern in $r's$. The q-pattern is a subsequence of $r's$, say $a = a_1 a_2 \cdots a_l$, whose smallest element is $a_i = \min_{1 \leq h \leq l} a_h$, corresponding to $q_i = 1$, and whose largest element is $a_j = \max_{1 \leq h \leq l} a_h$, corresponding to $q_j = l$. a_i may be in r' or in s.

Suppose a_i is in r'. Because every element of a is at least as large as a_1, while every element of s is smaller than a_1 (since a_1 is in r'), all of a must be in r'. That means r contains a q pattern, but that contradicts the assumption about r.

Suppose a_i is in s. Since $j > i$, a_j is also in s. As every element of a is no greater than a_j, none of them can be in r'. But then a is a subsequence of s, so s contains a q pattern. This contradicts the assumption about s.

Proof for Case 2. Here $q_i = l$ and $q_j = 1$ where $i < j$. Suppose we find a q-pattern in rs'. The q-pattern is a subsequence of rs', say $b = b_1 b_2 \cdots b_l$, whose largest element is $b_i = \max_{1 \leq h \leq l} b_h$, corresponding to $q_i = l$, and whose smallest element is $b_j = \min_{1 \leq h \leq l} b_h$, corresponding to $q_j = 1$. b_i may be in r or in s'.

Suppose b_i is in r. Because every element of b is not larger than b_1, while every element of s' is larger than b_1 (since b_1 is in r), all of b must be in r. That means r contains a q pattern, but that contradicts the assumption about r.

Suppose b_i is in s'. Since $j > i$, b_j is also in s'. As every element of b is no smaller than b_j, none of them can be in r. But then b is a subsequence of s', so s contains a q pattern. This contradicts the assumption about s.

Either way, we find a contradiction. Therefore, $\theta(r, s)$ is q-avoiding. So, we have an injection $\mathcal{S}_n(q) \times \mathcal{S}_m(q) \to \mathcal{S}_{n+m}(q)$, which proves that $S_n(q)S_m(q) \leq S_{n+m}(q)$. $\square$