Math 386 2011/12/14 **FINAL EXAM** Name SOLUTIONS

- Start each numbered problem on a *fresh page.*
- Hand in *both* this paper and test booklet.
- Show all your work for each problem; show enough work to fully justify your answer.
- Simplify answers as much as possible.
- All numerical answers may be in terms of actual numbers, factorials, and binomial coefficients *only*: not multinomial coefficients, for instance.
- Notation: $\mathcal{P}(S)$ is the power set of S.

(1) [Points: 10] Given k, what are the possible values of b and v in a $(b, v, k, k, 1)$-design?

Solution. Since $bk = rv$, we have $b = v$. Since $r(k-1) = \lambda(v-1)$, we have $k(k-1) = v - 1$. Therefore, $b = v = k^2 - k + 1$.

(2) [Points: 15] Solve the recurrence relation $a_n = 4a_{n-1} - 3^n$ (for $n \geq 1$), with $a_0 = 1$, by generating functions.

Solution. Let $A(x) := \sum_{n=0}^{\infty} a_n x^n$. Setting up the summation,

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} 4a_{n-1}x^n - \sum_{n=1}^{\infty} 3^n x^n,$$

thus (by setting $m = n - 1$ in the second summation),

$$A(x) - a_0 = 4x \sum_{m=0}^{\infty} a_m x^m - \Big[\sum_{n=0}^{\infty} 3^n x^n - 3^0\Big] = 4xA(x) - \Big[\frac{1}{1-3x} - 1\Big].$$

(Some people used $\sum_{n=1}^{\infty} 3^n x^n = 3x \sum_{n=0}^{\infty} 3^n x^n = \frac{3x}{1-3x}$ instead, which is perfectly good.) Moving all $A(x)$ terms to the left-hand side and substituting $a_0 = 1$,

$$A(x)[1-4x] = 1 - \frac{1}{1-3x} + 1 = 2 - \frac{1}{1-3x}.$$

(Some people made a sign error here, getting $1 - \cdots - 1$, which ruins the solution.) So,

$$A(x) = \frac{2}{1-4} - \frac{1}{(1-3x)(1-4x)}.$$

This is the first half of the solution.

In the second half, first you apply partial fractions to get $\frac{1}{(1-3x)(1-4x)} = 4\frac{1}{1-4x} - 3\frac{1}{1-3x}$, which you substitute into the formula for $A(x)$ to get

$$A(x) = \frac{2}{1-4x} - \Big[4\frac{1}{1-4x} - 3\frac{1}{1-3x}\Big] = 3\frac{1}{1-3x} - 2\frac{1}{1-4x}.$$

(Here you have to avoid another sign error. "Haste makes waste," said Mr. F.) Finally, expand the series:

$$A(x) = 3\sum_{n=0}^{\infty} 3^n x^n - 2\sum_{n=0}^{\infty} 4^n x^n = \sum_{n=0}^{\infty} [3 \cdot 3^n - 2 \cdot 4^n] x^n$$

so $a_n = 3^{n+1} - 2 \cdot 4^n$.

As a check, $a_0 = 3^1 - 2 \cdot 4^0 = 1$. Those who made sign errors may have gotten $a_n = 4^{n+1} - 3^{n+1}$, which gives the right value of a_0, unfortunately for you, but if you did check, I didn't deduct a point for not checking.

Some people defined $A(x) := \sum_{n=1}^{\infty} a_n x^n$. That works but you have to make some other modifications.

Some people first rewrote the recurrence as $a_{n+1} = 4a_n - 3^{n+1}$ (for $n \geq 0$). Remember, you're replacing n by $n+1$; if you don't follow through consistently, you ruin the problem.

(3) [Points: 15] A swan watches a line of n ducks swim by on the river. (On combinatorics tests, ducks always swim in a line in single file.) At some random point the swan grabs an odd number of consecutive ducks and carries them away to its secret castle under the bridge.

(a) What is the generating function for $d_n :=$ the number of ways the swan can do this?

Solution. [10 pts.] The main points are to use ordinary generating functions, to apply the product formula, to notice there are three subintervals in this problem, and to set up the generating function for odd sets.

The grabbees form a subinterval B in the line of ducks. They may have preceding ducks, forming a subinterval A of length ≥ 0, and following ducks, forming a subinterval C of length ≥ 0 in the line. Since we're doing intervals we use ordinary generating functions. Let $D(x) = \sum_{n=0}^{\infty} d_n x^n$ be the generating function of the sequence $(d_n)_{n=0}^{\infty}$. Let $A(x)$, $B(x)$, and $C(x)$ be the generating functions of the number of ways to form intervals A, B, and C, respectively. Then $A(x) = C(x) = \sum_{n=0}^{\infty} 1x^n = 1/(1-x)$. Also, $B(x) = \sum_{n=0}^{\infty} b_n x^n$ where $b_n = 0$ if n is even and 1 if n is odd, so $B(x) = \sum_{k=0}^{\infty} 1x^{2k+1} = x/(1-x^2)$. By the product formula for generating functions,

$$D(x) = A(x)B(x)C(x) = \frac{x}{(1-x)^2(1-x^2)}.$$

Some people used only two intervals, getting a formula like $D(x) = A(x)B(x) = x/(1-x)(1-x^2) = x/(1-x)^2(1+x)$. This got partial credit.

(b) Find a closed formula for d_n.

Solution. [5 pts.] At this point you can use partial fractions or a trick. I'll do partial fractions.

$$\frac{x}{(1-x)^2(1-x^2)} = \frac{x}{(1-x)^3(1+x)} = \frac{\alpha}{1-x} + \frac{\beta}{(1-x)^2} + \frac{\gamma}{(1-x)^3} + \frac{\delta}{1+x}.$$

(You cannot use $\gamma/(1-x^2)$ instead of the last two terms. It gives a false answer.) The solution comes from analyzing

$$x = \alpha(1-x)^2(1+x) + \beta(1-x)(1+x) + \gamma(1+x) + \delta(1-x)^3,$$

and we get $\alpha = -1/8$, $\beta = -1/4$, $\gamma = 1/2$, and $\delta = -1/8$. Therefore,

$$\begin{aligned} D(x) &= -\frac{1}{8}\frac{1}{1-x} - \frac{1}{4}\frac{1}{(1-x)^2} + \frac{1}{2}\frac{1}{(1-x)^3} - \frac{1}{8}\frac{1}{1+x} \\ &= -\frac{1}{8}\sum_{n=0}^{\infty} x^n - \frac{1}{4}\sum_{n=0}^{\infty}\binom{n+1}{1}x^n + \frac{1}{2}\binom{n+2}{2}\sum_{n=0}^{\infty} x^n - \frac{1}{8}\sum_{n=0}^{\infty}(-x)^n \\ &= \sum_{n=0}^{\infty}\Big[-\frac{1}{8} - \frac{n+1}{4} + \frac{1}{2}\binom{n+2}{2} - \frac{(-1)^n}{8}\Big]x^n. \end{aligned}$$

The coefficient of x^n is d_n so the solution is

$$d_n = -\frac{1}{8} - \frac{n+1}{4} + \frac{1}{2}\binom{n+2}{2} - \frac{(-1)^n}{8} = \frac{2n^2+4n+1-(-1)^n}{8}.$$

As a simple check, $d_0 = 0$ and $d_1 = 1$, which are both obviously correct numbers. I also checked d_2 up to d_5, which are easy to do by hand, and the formula works. Yay!

(4) [Points: 15] Given that $F(x) = e^{x^2}$ is the exponential generating function of a sequence $(a_n)_{n=0}^{\infty}$, find the values of the numbers a_n. (Be careful.)

Solution. First step: Use the series for e^y to get

$$F(x) = \sum_{n=0}^{\infty}\frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty}\frac{x^{2n}}{n!}.$$

Then you compare this series with the exponential generating function

$$F(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

This is where you have to be very careful: The n's in both formulas are not the same n. You can see that if you compare the exponents of x. We have to rewrite the first formula with x^n, or actually it's simpler if we use index m, so in the first formula $m = 2n$ (which is even) and in the second formula $m = n$. We get

$$\sum_{nm=0}^{\infty} a_m \frac{x^m}{m!} = F(x) = \sum_{n=0}^{\infty}\frac{x^{2n}}{n!} = \sum_{m=0,\text{ even}}^{\infty}\frac{x^m}{(m/2)!}.$$

Two power series are equal if and only if the coefficients of each power of x are equal; therefore,

$$a_m\frac{1}{m!} = \begin{cases} \frac{1}{(m/2)!} & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd,} \end{cases}$$

which gives the solution

$$a_m = \begin{cases} \frac{m!}{(m/2)!} & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

(It looks nicer if you write $m = 2k$ and $a_{2k} = (2k)!/k! = (2k)_k$ in the even case.)

A *serious error* is to have x in your "coefficient" a_m. If you think the coefficient of $x^n/n!$ in $\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$ is x^n, you haven't understood what we're doing.

(5) [Points: 15] Evaluate $S_n(123, 132)$, the number of n-permutations that avoid 123 and 132.

Solution. Consider the location of n, say the kth position where $1 \le k \le n$, so $p_k = n$. The sequence before n cannot have any ascent, because that would form a 123 pattern with n. Thus, before n the sequence is decreasing. The numbers before n must be bigger than those after n, or we would have a 132 pattern (with n as the 2). Thus, the permutation before n is $n-1, n-2, \ldots, n-k+1$ (note that there are $k-1$ numbers here) and the part after n is a permutation of $[n-k]$ which avoids 123 and 132.

Now, let $a_n = S_n(123, 132)$, for short. We just proved that $a_n = \sum k = 1^n a_{n-k} = a_0 + a_1 + a_2 + \cdots + a_{n-1}$. Since obviously $a_0 = 1 = a_1$ and $a_2 = 2$, we can guess $a_n = 2^{n-1}$ (if $n > 0$) and prove it by the inductive argument that $a_n = 1 + (1 + 2 + 2^2 + \cdots + 2^{n-2}) = 2^{n-1}$.

There are solutions that give the formula 2^{n-1} more directly, but I'll stop here.

(6) [Points: 10] How many permutations of $[n]$ have exactly $n-1$ left-to-right minima?

Solution. I'll give two solutions.

Solution I. (Clever.) A permutation p in one-line form corresponds to a permutation q in canonical cycle form where the left-right minima of p are where q begins a new cycle. Thus, the number of left-right minima of p equals the number of cycles in q. In this case q has $n-1$ cycles, so all cycles are singletons except for one 2-cycle. The number of such permutations equals the number of ways to chooe the two numbers for the 2-cycle, which is $\binom{n}{2}$, so the answer is $\binom{n}{2}$.

Solution II. (Direct.) If we choose one element $i \in [n]$ to be the one that is not the left-right minimum, then all the other numbers must be decreasing. That forces everything except where we put i. We must put i after a smaller number. The smaller numbers are $i+1, \ldots, n$, of which there are $n-i$, so there are $n-i$ places to put i. Thus, the number of permutations is

$$\sum_{i=1}^{n} (n-i) = 1 + 2 + \cdots + (n-1) = \binom{n}{2}.$$

(7) [Points: 10] How many permutations of $[n]$ have 1 and 2 together in a cycle of length 5? Assume $n \ge 5$.

Solution. I'll give two solutions.

Solution I. (Direct.) Choose the 3 other numbers to go in the cycle with 1 and 2: there are $\binom{n-2}{2}$ ways to do so. There are $(5-1)! = 4!$ ways to arrange these 5 numbers in a cycle. There are $(n-5)!$ ways to permute the remaining $n-5$ numbers of $[n]$. Thus, the answer is

$$\binom{n-2}{2} 4!(n-5)!.$$

This simplifies to $4(n-2)!$, but that's not required.

Solution II. (Clever.) Think of how this permutation is represented in canonical cycle form. Converted to one-line notation, it is $q_1 \cdots q_{n-4} \cdots q_n$. The last 5 elements are the 5-cycle with 1 in it; since 1 is smallest in its cycle, $q_{n-4} = 1$. Then one of $q_{n-3}, \ldots, q_n$ equals 2; there are 4 choices here. Finally, there are $n-2$ empty spots that can be filled by any of the $(n-2)!$ permutations of $3, 4, \ldots, n$. The final answer is $4(n-2)!$.

(8) [Points: 15] Give a combinatorial proof that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ for $n \geq k > 0$.

Solution. The LHS is the number of ways to take a k-element subset from $[n]$. Each such subset contains n or not. If it contains n, we get it by choosing n and then $k-1$ elements from $[n] \setminus n$, which can be done in $\binom{n-1}{k-1}$ ways. If it doesn't contain n, we get it by choosing $k-1$ elements from $[n] \setminus n$, which can be done in $\binom{n-1}{k}$ ways. The RHS is the sum of these, which counts all ways of choosing k elements from $[n]$, the same as the LHS. Therefore, the LHS = the RHS.

(9) [Points: 15] How many numbers are there, from $30,000$ to $999,999$, with all digits distinct and in increasing order?

Solution. This tests whether you can find the combinatorics in an unfamiliar problem.

Each number is determined by its set of digits since there's only one way to put them in order. So, for 5-digit numbers, you choose a 5-subset of $\{3, 4, \ldots, 9\}$, which can be done in $\binom{7}{5}$ ways, and for 6-digit numbers (whose leading digit cannot be 0) you choose a 6-combination from $[9]$, which can be done in $\binom{9}{6}$ ways. The answer is therefore

$$\binom{7}{5} + \binom{9}{6} = \binom{7}{2} + \binom{9}{3} = 105.$$

(Any of these expressions is correct.)

(10) [Points: 15] How many northeastern lattice paths are there from (0,0) to (6,18), but not passing through the flood at location (2,7)?

Solution. This tests whether you know the formula for lattice paths, and also whether you can use it in a slightly complicated problem.

The number of paths from $(0,0)$ to (m,n) is $\binom{m+n}{m}$ or $\binom{m+n}{n}$. Thus there are $\binom{24}{6}$ paths from (0,0) to (6,18). We need to subtract the number of them that go through (2,7). That number equals the number of paths from (0,0) to (2,7) to (6,18), which is the product $\binom{9}{2}\binom{15}{4}$. Thus, the answer is $\binom{24}{6} - \binom{9}{2}\binom{15}{4}$.

(11) [Points: 15] In the affine plane $\mathrm{AP}(\mathbb{Z}_3)$:

(a) Set up a table of the y values of all points (x, y) on lines with slope 2. [Your table should have one row for each x value and one column for each line.]

Solution. [10 pts.] One thing I'm looking for is that you can do arithmetic in $\mathbb{Z}_3$. There are three lines of the form $y = 2x + b$ with $b = 0, 1, 2$. The table is

x	$b=0$ $y=2x$	$b=1$ $y=2x+1$	$b=2$ $y=2x+2$
0	0	1	2
1	2	0	1
2	1	2	0

(b) Explain why this table is a Latin square. A rigorous proof gets full credit, but an intuitive explanation can get partial credit.

Solution. [5 pts.] The table is a Latin square because it has three symbols, 0, 1, and 2, and each one appears exactly once in each row and each column. (In other words, explain the properties that make it a Latin square.)

N.B. I meant to ask you to prove that any such table for any $\mathbb{Z}_p$ (p a prime number) is a Latin square, but that's not what it says! So, I didn't expect that in the grading. However, for bonus credit up to 6 pts., here is a good proof. (A few people got some bonus credit.)

Proof. There is no duplication in a row because as you go across a row you add 1 at each column, giving all the values $0, 1, 2$ of $\mathbb{Z}_3$ (in general, $0, 1, \ldots, p-1$ of $\mathbb{Z}_p$) in cyclic order. (A more precise version of this: In the row of x, equal y values $y_1 = y_2$ in the columns of b_1 and b_2 mean $mx + b_1 = mx + b_2$, which implies $b_1 = b_2$. But then the two columns are the same column.)

There is no duplication in a column because $m = 2$ is invertible in $\mathbb{Z}_2$ (generally, because m is invertible in $\mathbb{Z}_p$—which is true because $m \neq 0$ and p is prime; this is number theory). Specifically, if in column b there are equal y values, $y_1 = y_2$ in rows x_1 and x_2, then $mx_1 + b = mx_2 + b$ so $mx_1 = mx_2$. Since m^{-1} exists, this implies $x_1 = x_2$, so were were in only one row, not two different rows.