Rooks in the Staircase
(Chapter 5, Exercise 4)
Math 386, December 1, 2011

The problem in (b) is to count the number of non-attacking placements of $k$ rooks in the staircase Ferrers diagram, which is the diagram of the partition $(n-1, n-2, \ldots, 2, 1)$ of $\binom{n}{2}$.

A couple of beginning examples:

$$h(0) = 1, \quad h(1) = \binom{n}{2},$$

since with $k = 0$ we don't do anything, and with $k = 1$ we put a rook anywhere. The pattern is obvious, right? Not yet! I think it's easier to find a general solution than to infer a pattern.

Part (a) of the question suggests the answer to (b) is either $S(n, k)$ or $S(n, n-k)$, since the answer to part (a), the Bell number $B_n$, is the sum of all Stirling numbers $S(n, i)$ for fixed $n$. That guided me in thinking about part (b).

**Proposition 0.1.** [{P:k-rooks} ] *The number of ways to place $k$ nonattacking rooks in the staircase diagram is $S(n, n-k)$.*

*Proof.* We set up a function $\theta : \mathcal{R}_k \to \Pi_{n,n-k}$, where $\mathcal{R}_k$ is the set of all possible arrangements of $k$ rooks in the diagram and $\Pi_{n,n-k}$ is the set of all partitions of $[n]$ into $n-k$ parts. We also set up a function $\varphi : \Pi_{n,n-k} \to \mathcal{R}_k$, and we'll show they are inverse functions. Therefore, they are bijections, so $|\mathcal{R}_k| = |\Pi_{n,n-k}| = S(n, n-k)$.

The secret trick is to number the rows $1, 2, \ldots, n-1$ from top to bottom and number the columns $n, n-1, \ldots, 2$ from left to right. This sets up a coordinate system in the diagram. Every square $(x, y)$ satisfies $1 \leq x < y \leq n$.

An arrangement of rooks is a set of coordinates of squares, i.e., for $k$ rooks we give $k$ coordinate pairs. Each rook has coordinates $(x, y)$ that satisfy $1 \leq x < y \leq n$.

Now let's define $\theta$. Let $\mathbf{R}$ be an arrangement of $k$ rooks. I will define the partition $\theta(\mathbf{R})$. To construct the partition, put two numbers $a, b \in [n]$, with $a < b$, into the same block of the partition if (but not only if!) there is a rook in $\mathbf{R}$ with coordinates $(a, b)$. This tells us about some pairs, and that implies the partition in which those pairs are in the same block.

For example, suppose $n = 6$, $k = 3$, and the three rooks are in positions $(1, 6)$ (the top left square), $(2, 4)$, and $(4, 5)$; that is, $\mathbf{R} = \{(1, 6), (2, 4), (4, 5)\}$. Then $1, 6$ are in the same block, and $2, 4, 5$ are in the same block. The partition is $\{16, 245, 3\}$.

Now let's define $\varphi$. Let $\pi = \{B_1, B_2, \ldots, B_k l\}$ be a partition of $[n]$. For each block $B$, arrange its elements in increasing order, say $a_1 < a_2 < \cdots < a_m$ where $m = |B|$. (That's the second secret trick.) Then $(a_1, a_2), (a_2, a_3), \ldots, (a_{m-1}, a_m)$ will be in the rook arrangement $\varphi(\pi)$. Do this for each block and you have $\varphi(\pi)$.

For example, if $n = 7$ and $\pi = \{134, 27, 5, 6\}$, then $\varphi(\pi) = \{(1, 3), (3, 4), (2, 7)\}$.

We need to prove these are valid functions with the codomains I claimed for them. First, if we start with $\pi \in \Pi_{n,n-k}$, we certainly get a well-defined rook arrangement. The number of rooks is $|B_i| - 1$ for each block, so it $= \sum_{i=1}^{k}(|B_i| - 1) = \sum_{i=1}^{k} |B_i| - (n-)k = k$. Thus, $\varphi(\pi) \in \mathcal{R}_k$. So, $\varphi$ is really a function from $\Pi_{n,n-k}$ to $\mathcal{R}_k$.

If we have $\mathbf{R} \in \mathcal{R}_k$, we need to prove that $\theta(\mathbf{R})$ is a partition of $[n]$ with $n-k$ blocks. We need to study the structure of a rook arrangement. Suppose it has a rook in position $(a, b)$. Then no other rook in $\mathbf{R}$ can be in row $a$ or column $b$. Therefore, the only way $a$ can be in a second coordinate pair is to be the $y$-coordinate, i.e., if there is a rook at $(z, a)$ for some $z < a$. Similarly, the only way $b$ can be in another coordinate pair is to be the $x$-coordinate, i.e., if there is a rook at $(b, c)$ for some $c > b$. Furthermore, $a$ and $b$ can only be in at most two coordinate pairs of rooks in $\mathbf{R}$. It follows that, if we chain the pairs in $\mathbf{R}$ together as follows: $(a, b), (b, c), \ldots, (e, f)$, then $a < b < c < \cdots < e < f$ and all these numbers are in the same block of $\theta(\mathbf{R})$. Another block might be derived from $(a', b'), (c', d'), \ldots, (e', f')$, where $a' < b' < \ldots < e' < f'$ and they are all different from $a, b, c, \ldots$; so we get a separate block of $\theta(\mathbf{R})$ containing $a', b', c', \ldots, e', f'$. Therefore, the blocks of $\theta(\mathbf{R})$ are obtained by chaining together rook coordinates. How many blocks are there? Start with every number of $[n]$ in a separate block. The first rook combines two numbers in a block. The second rook combines two blocks into one, and so on. We combine $k$ times, so we have $n - k$ separate blocks when done. Therefore, $\theta(\mathbf{R}) \in \Pi_{n,n-k}$, which is the right set to make $\theta$ well defined.

Now that we know $\varphi$ and $\theta$ are functions with the right domains and codomains, we need to show they are inverses. Suppose we take $\theta(\varphi(\mathbf{R}))$. Look at what happens to a chained series of coordinate pairs, $(a, b), (b, c), \ldots, (e, f)$. Applying $\theta$, it becomes a block $\{a, b, c, \ldots, e, f\}$ with $a < b < c < \cdots < e < f$. Applying $\varphi$ to this block we get the original coordinate pairs. Therefore, $\theta(\varphi(\mathbf{R})) = \mathbf{R}$.

Take $\varphi(\theta(\pi))$. If $B \in \pi$, say $B = \{a, b, c, \ldots, e, f\}$ with $a < b < c < \cdots < e < f$, then $\theta(\pi)$ has the coordinate pairs $(a, b), (b, c), \ldots, (e, f)$, but no other pairs involving the numbers $a, b, \ldots, f$. So, when we apply $\varphi$ to $\theta((\pi)$, the coordinates $a, b, c, \ldots, e, f$ will be together in a block, but no other numbers will be in the same block. That is, $\varphi(\theta(\pi)) = \pi$.

Since we proved there are inverse functions between $\mathcal{R}_k$ and $\Pi_{n,n-k}$, those functions are bijections; therefore $|\mathcal{R}_k| = |\Pi_{n,n-k}|$. □