Subdesigns
(Solution of Ch. 18, Problem 40)
December 10, 2011

We have a (v, k, λ)-design $\mathcal{D} = (S, \mathcal{B}_{\mathcal{D}})$ and a (w, k, λ)-design $\mathcal{F} = (Q, \mathcal{B}_{\mathcal{F}})$ such that $Q \subseteq S$ and $\mathcal{B}_{\mathcal{F}} \subseteq \mathcal{B}_{\mathcal{D}}$ and $w < v$. We're supposed to prove that $w \leq \lambda(v-1)/(k-1)$. We may assume $k \geq 2$, as otherwise we'd be dividing by zero.

The fact that every block of $\mathcal{F}$ is a block of $\mathcal{D}$ does not necessarily mean (to me) that it has the same points as a block in both designs. However, since the block size in $\mathcal{F}$ is the same as that in $\mathcal{D}$, it seems obvious that a block of $\mathcal{F}$ has the same points in $\mathcal{F}$ as it does in $\mathcal{D}$.

Lemma 1. [{L:subdesign}] *There are three kinds of block of $\mathcal{D}$.*

(0) Blocks disjoint from Q.

(1) Blocks that contain just one point of Q.

(2) Blocks that are contained in Q.

Furthermore, $\mathcal{B}_{\mathcal{F}} = \{B \in \mathcal{B}_{\mathcal{D}} : B \subseteq Q\}$.

Proof. For the first part, I should prove that a block (of $\mathcal{D}$) that contains more than one point of Q is a subset of Q. Assume B is such a block and let $q_1, q_2 \in B \cap Q$. Since there are λ blocks of $\mathcal{D}$ on $\{q_1, q_2\}$ and there are also λ blocks of $\mathcal{F}$ on $\{q_1, q_2\}$, every block of $\mathcal{D}$ that contains $\{q_1, q_2\}$ must be a block of $\mathcal{F}$. A block of $\mathcal{F}$ is necessarily a subset of Q. That proves the first part.

In particular, $B \in \mathcal{B}_{\mathcal{F}}$. Therefore, every block of type (2) is a block of $\mathcal{F}$; that is, $\mathcal{B}_{\mathcal{F}} \supseteq \{B \in \mathcal{B}_{\mathcal{D}} : B \subseteq Q\}$. Since every block of $\mathcal{F}$ is contained in Q, we have the second part. □

Now we count the blocks on a point $y \notin Q$. (Such a point y exists because $v > w$.) Let's count the pairs (q, B) such that $q \in Q$ and B is a block that contains both q and y. (Such a block must be of type (1).)

First, if we pick q there are λ blocks that contain both q and y. There are w ways to pick q so there are $w\lambda$ pairs (q, B).

Second, we can pick B, a block of type (1) containing y, first, and then pick the point $q \in B \cap Q$. Let r_1 be the number of ways to pick B; there are r_1 pairs because there's only one $q \in B \cap Q$. Thus, there are r_1 pairs (q, B).

Now, $r_1 \leq r$, the total number of blocks that contain y. So, $w\lambda = r_1 \leq r = \lambda(v-1)/(k-1)$. Dividing by λ, we obtain the inequality

$$w \leq \frac{v-1}{k-1}.$$

Q.E.D.