Stirling's Approximation and Derangement Numbers

First, Stirling's approximation for $n!$; then binomial coefficients, then D_n.

1. Stirling's Approximation

1.1. **Stirling's Approximation to the Factorial.**

Stirling's approximation is

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}. \tag{1}$$

This is the simplest approximation used when you only need a good estimate. The precise meaning of $\approx$ in (1) is that the *quotient* of these two quantities approaches 1:

$$\lim_{n\to\infty} \frac{n!}{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}} = 1.$$

The difference is a different story; it gets large. A simple estimate of the difference is:

$$n! - \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\, \frac{1}{12n}. \tag{2}$$

Written as an approximation of $n!$:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \left[1 + \frac{1}{12n}\right]. \tag{3}$$

The exact infinite series is:

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \left[1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{52840n^3} - \frac{571}{2488320n^4} + \cdots\right]. \tag{4}$$

This series does not converge. You may wonder what good a non-convergent series is. Thats a good question. If you take a certain number of terms of the series, you get a very good approximation to $n!$. If you take too many terms, you get a terrible approximation. The best number of terms to take depends on n; the bigger n is, the more terms you need for the best approximation. This gets very complicated, so mostly we just use (1).

1.2. **Approximating Binomial Coefficients and Catalan Numbers.**

As one of the more useful applications, let's approximate $\binom{2n}{n}$, the middle binomial coefficient (in row $2n$ of Pascal's triangle).

$$\binom{2n}{n} = \frac{(2n)!}{n!^2} \approx \frac{(2n/e)^{2n} \cdot \sqrt{2\pi \cdot 2n}}{\left[(n/e)^n \cdot \sqrt{2\pi n}\,\right]^2} = \frac{4^n (n/e)^{2n} \cdot 2\sqrt{\pi n}}{(n/e)^{2n} \cdot 2\pi n} = \frac{4^n}{\sqrt{\pi n}}. \tag{5}$$

This is a remarkably simple formula, even for a crude estimate. Applying it to the Catalan number C_n,

$$C_n = \frac{1}{n+1}\binom{2n}{n} \approx \frac{4^n}{n\sqrt{\pi n}}. \tag{6}$$

Equation (5) gives an approximation to $\binom{m}{\frac{1}{2}m}$ (when m is even). Let's try this for other fractions. Say, $\binom{m}{k}$ where $k \approx \frac{p}{q}m$ for a fixed proportion $\frac{p}{q}$ such that $0 < \frac{p}{q} < 1$ (i.e., $0 < p < q$).

$$\begin{aligned}
\binom{m}{k} = \frac{m!}{k!(m-k)!} &\approx \frac{(m/e)^m \cdot \sqrt{2\pi m}}{\left[(k/e)^k \cdot \sqrt{2\pi k}\right]\left[((m-k)/e)^{m-k} \cdot \sqrt{2\pi(m-k)}\right]} \\
&\approx \frac{(m/e)^m \cdot \sqrt{2\pi m}}{(\frac{p}{q}m/e)^{\frac{p}{q}m}(\frac{q-p}{q}m/e)^{\frac{q-p}{q}m} \cdot 2\pi m\sqrt{\frac{p}{q}\frac{q-p}{q}}} \\
&= \frac{(m/e)^m}{\left[(\frac{p}{q})^{\frac{p}{q}}(\frac{q-p}{q})^{\frac{q-p}{q}}\right]^m (m/e)^{\frac{p}{q}m+\frac{q-p}{q}m} \cdot \sqrt{2\pi m}\sqrt{\frac{p}{q}\frac{q-p}{q}}} \\
&= \left[\left(\frac{q}{p}\right)^p \left(\frac{q}{q-p}\right)^{q-p}\right]^{m/q} \frac{1}{\sqrt{2\pi m}} \frac{q}{\sqrt{p(q-p)}}
\end{aligned}$$

because $\frac{p}{q}m + \frac{q-p}{q}m = m$,

$$= \left[\frac{q^q}{p^p(q-p)^{q-p}}\right]^{m/q} \frac{q}{\sqrt{p(q-p)}} \frac{1}{\sqrt{2\pi m}}.$$

Summarizing,

$$\binom{m}{k} \approx \left[\frac{q}{[p^p(q-p)^{q-p}]^{1/q}}\right]^m \frac{q}{\sqrt{p(q-p)}} \frac{1}{\sqrt{2\pi m}}. \tag{7}$$

For instance, if $k \approx \frac{1}{2}m$, so that $\frac{p}{q} = \frac{1}{2}$, we get the approximation

$$\binom{m}{\frac{1}{2}m} \approx \left[\frac{2}{[1^1(2-1)^{2-1}]^{1/2}}\right]^m \frac{2}{\sqrt{1(2-1)}} \frac{1}{\sqrt{2\pi m}} = 2^m \frac{1}{\sqrt{\pi(m/2)}},$$

which agrees with Equation (5). If $k \approx \frac{1}{3}m$, we get

$$\binom{m}{\frac{1}{3}m} \approx \left[\frac{3}{[1^1(3-1)^{3-1}]^{1/3}}\right]^m \frac{3}{\sqrt{1(3-1)}} \frac{1}{\sqrt{2\pi m}} = \left[\frac{3}{\sqrt[3]{4}}\right]^m \frac{3}{2\sqrt{\pi m}}.$$

Finally, if $k \approx \frac{1}{4}m$, then

$$\binom{m}{\frac{1}{4}m} \approx \left[\frac{4}{[1^1(4-1)^{4-1}]^{1/4}}\right]^m \frac{4}{\sqrt{1(4-1)}} \frac{1}{\sqrt{2\pi m}} = \left[\frac{4}{3^{3/4}}\right]^m \frac{4}{\sqrt{6\pi m}}.$$

2. Approximations of the Derangement Numbers

2.1. **Calculus-Based Approximation to the Derangement Numbers.**

This is the amazingly simple formula (I know of no similar examples):

$$D_n = [[n!/e]], \tag{8}$$

where [[ ]] means take the nearest integer. To prove this we take the factorial formula for D_n:

$$D_n = n! \sum_{j=0}^{n} (-1)^j \frac{1}{j!} \tag{9}$$

and compare it to the power series for e^x with $x = -1$:

$$e^{-1} = \sum_{j=0}^{\infty} (-1)^j \frac{1}{j!}.$$

We see that $D_n/n!$ is simply the n-th Taylor polynomial of e^{-1}. Furthermore, the series for e^{-1} satisfies the requirement of the Alternating Series Test: The terms alternate in sign, they are decreasing in absolute value, and they approach 0. Therefore, we can apply the Alternating Series Error Test: the error if we stop at term n is less than the (absolute value of the) next term. The error is $|e^{-1} - D_n/n!|$. The (absolute value of the) next term is $1/(n+1)!$, so

$$\left| \frac{D_n}{n!} - \frac{1}{e} \right| < \frac{1}{(n+1)!}.$$

Now, multiply by $n!$; this gives

$$\left| D_n - \frac{n!}{e} \right| < \frac{1}{n+1}.$$

If $n \geq 1$, then $1/(n+1) \leq 1/2$; therefore,

$$\left| D_n - \frac{n!}{e} \right| < \frac{1}{2}.$$

There can be only one integer within $< 1/2$ of a real number, so D_n is indeed the nearest integer to $n!/e$. We have proved (8) for $n > 0$. It also happens to be true for $n = 0$, so (8) is valid for every n such that D_n is defined.

2.2. **Stirling's Approximation to the Derangement Number.**

From (1) and (8) (rewritten as $D_n \approx n!/e$) we get

$$D_n \approx \frac{n^n}{e^{n+1}} \sqrt{2\pi n}. \tag{10}$$

The precise meaning of $\approx$ in (10) is that the quotient approaches 1:

$$\lim_{n\to\infty} D_n \Big/ \frac{n^n}{e^{n+1}} \sqrt{2\pi n} = 1.$$

How good is this approximation? It's not clear. Let's look at the data. We calculate for small values of n:

n	$n!/e$	D_n	Stirling approximation
0	.37	1	0
1	.37	0	$\sqrt{2\pi}/e^2 \approx 0.12$
2	.74	1	$4\sqrt{4\pi}/e^3 \approx 0.71$
3	2.21	2	$27\sqrt{6\pi}/e^4 \approx 2.15$
4	8.83	9	$256\sqrt{8\pi}/e^5 \approx 8.65$
5	44.15	44	$3125\sqrt{10\pi}/e^6 \approx 43.42$
6	264.87	265	$46656\sqrt{12\pi}/e^7 \approx 261.22$
7	1854.1	1854	$823543\sqrt{14\pi}/e^8 \approx 1832.19$
8	14832.9	14832	$16777216\sqrt{16\pi}/e^9 \approx 14679.27$

Conclusion: We don't get nearly as accurate an approximation from Stirling's formula as with $n!/e$. That shows how exceptionally good an approximation $n!/e$ is; it's much closer than most approximations. Stirling's approximation to D_n is more typical: the absolute error gets larger but the *proportional error* gets smaller as n increases.