Math 386 2012/11/20 **TEST II** Name SOLUTIONS

- Start each numbered problem on a *fresh page.*
- Hand in *both* this paper and test booklet.
- Show all your work for each problem; show enough work to fully justify your answer.
- Simplify answers as much as possible.
- All numerical answers may be in terms of actual numbers, factorials, falling factorials, and binomial coefficients; but *not* multinomial coefficients, for instance.

(1) [Points: 2] Did you read the instructions and will you follow them? Ans.___________

(2) [Points: 15] Use *generating functions* to find an explicit formula for a_n, $n \geq 0$, given that $a_n = 3a_{n-1} + 2^n$ for $n > 0$ and $a_0 = 1$.

Solution. Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$, the OGF of a_n. Then

$$A(x) = \frac{1}{(1-2x)(1-3x)} = -\frac{2}{1-2x} + \frac{3}{1-3x} = \sum_{n=0}^{\infty} \big[-2(2^n) + 3(3^n) \big] x^n,$$

so $a_n = -2^{n+1} + 3^{n+1}$. Quick check: The formula gives $a_0 = -2^1 + 3^1 = 1$, agreeing with the initial condition stated in the problem.

(3) [Points: 12] Find a closed formula for the ordinary generating function of the numbers h_n, where h_n is the number of compositions of n into parts that are either 2 or 5. (Don't solve for h_n.)

Solution. Let $H(x)$ be the OGF of h_n. Thinking of $[n]$ as an interval of length n, we want to split it up into any number of subintervals of lengths 2 and 5. That will represent a unique composition whose parts (corresponding to the subintervals) are equal to 2 or 5. Let a_k be the number of ways we can have an interval of length k in a composition. Then $a_2 = a_5 = 1$, but all other $a_k = 0$. The OGF of a_k is $A(x) = 1x^2 + 1x^5$. The general theory says

$$H(x) = \frac{1}{1-A(x)} = \frac{1}{1-x^2-x^5}.$$

(4) [Points: 10] Count the inversions in the permutation and find the parity of the permutation.

(a) $p = 412356$

Solution. The inversions are 41, 42, 43. There are 3 of them, so p is odd.

(b) $q = (412)(356)$

Solution. We first have to put q into one-line form; that is, $q = 245163$. The number of inversions that begin with 1 is 0, with 2 is 1, with 3 is 0, with 4 is 2, with 5 is 2, with 6 is 1. The total number of inversions is 6, so q is even.

(5) [Points: 12] Prove that the number of odd permutations of $[n]$ equals the number of even permutations of $[n]$.

Solution. First proof. Let's define $\mathrm{Perm}_e(n)$ to be the set of even permutations and $\mathrm{Perm}_o(n)$ to be the set of odd permutations. We define a function $\theta : \mathrm{Perm}_e(n) \to \mathrm{Perm}(n)$ defined by $\theta(p) = p$ with 1 and 2 interchanged. This changes the number of inversions by 1 (increase or decrease), because if we consider any pair $p_i p_j$ where $i < j$, it remains in the same order in $\theta(p)$ in every case except when $\{p_i, p_j\} = \{1, 2\}$. This proves that $\theta(p)$ is odd. The same rule defines a function $\theta' : \mathrm{Perm}_o(n) \to \mathrm{Perm}_e(n)$. It's clear that θ' is the inverse function of θ. Therefore, θ is a bijection, hence $\mathrm{Perm}_e(n)| = | \mathrm{Perm}_o(n)$.

Second proof. The setup is the same, but I define $\theta(p) = p_2 p_1 p_3 \cdots p_n$, where $p = p_1 p_2 p_3 \cdots p_n$, i.e., interchange the first two elements of p.

(6) [Points: 10] Let g be the function (as described in the book) that converts a permutation of $[n]$ with k cycles into a permutation of $[n]$ with k left-to-right maxima. Find $g(p)$ where $p = (412)(356)$.

Solution. First we have to express p in canonical cycle form: $p = (412)(635)$. Then we erase the parentheses, getting $g(p) = 412635$.

(7) [Points: 12] Find the value of $s(n, 2)$ for $n \geq 1$, where $s(n, k)$ denotes the Stirling number of the first kind.

Solution. First method. We know $s(n, 2) = (-1)^{n-2} c(n, 2) = (-1)^n c(n, 2)$ where $c(n, 2)$ is the number of permutations of $[n]$ with exactly two cycles. We need a formula for $c(n, 2)$.

To get two cycles, first we choose a nonempty subset $S \subset [n]$ and its complement, $T = [n] \setminus S$. Then we turn S and T into cycles. Since the number of ways to do that depends on $|S|$ and $|T|$, we organize this computation by $k = |S|$. We choose $0 < k < n$. Then we choose S of size k; there are $\binom{n}{k}$ ways to do that. Then we convert S into a cycle; there are $(k-1)!$ ways to do so; and we convert T into a cycle; there are $(n-k-1)!$ ways to do that. Finally, we sum over k, getting $\sum_{k=1}^{n-1} \binom{n}{k} (k-1)!(n-k-1)!$.

However, we have to divide by 2, because we counted the same pair of sets, $\{S, T\}$, twice, once for S and once for T. So,

$$c(n, k) = \frac{1}{2} \sum_{k=1}^{n-1} \binom{n}{k} (k-1)!(n-k-1)!.$$

The final answer, after simplification, is

$$s(n, 2) = (-1)^n \frac{n!}{2} \sum_{k=1}^{n-1} \frac{1}{(k-1)(n-k-1)}.$$

Solution. Second method. Start the same way. To avoid duplication, we choose S to contain 1. Then, for each k in $0 < k < n$, there are $\binom{n-1}{k-1}$ choices for S. Counting the number of ways to make S and T into cycles, we get

$$c(n,k) = \sum_{k=1}^{n-1} \binom{n-1}{k-1}(k-1)!(n-k-1)! = (n-1)! \sum_{k=1}^{n-1} \frac{1}{n-k-1}.$$

If we replace $j = n-k$ in the last summation, we get

$$s(n,2) = (-1)^n (n-1)! \sum_{j=1}^{n-1} \frac{1}{j} = (-1)^n (n-1)! \Big[\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n-1}\Big].$$

The methods seem to give different answers. I infer that there's an equation that shows they are not different. I know what it is! Do you?

Other possible answers. For instance,

$$s(n,2) = \begin{cases} (-1)^n n! \displaystyle\sum_{k=1}^{(n-1)/2} \frac{1}{(k-1)(n-k-1)} & \text{if } n \text{ is odd,} \\ (-1)^n n! \Bigg[\displaystyle\sum_{k=1}^{(n-1)/2} \frac{1}{(k-1)(n-k-1)} + \frac{1}{2(\frac{n}{2}-1)^2}\Bigg] & \text{if } n \text{ is even.} \end{cases}$$

(Some people found answers like this, but with binomial coefficients instead of all my simplifications.)

(8) [Points: 12] Find the number of permutations p of $[10]$ such that $p(i) \neq i$ for all even numbers $i \in [10]$.

Solution. This is a task for PIE. We have the universe $U = \text{Perm}(10)$. We want to avoid the properties $p(i) = i$, for every even $i \in [10]$. Thus, let $A_i = \{p \in U : p_{2i} = 2i\}$ for $i = 1, 2, 3, 4, 5$. We want the value of

$$\begin{aligned} &\big|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4 \cap \bar{A}_5\big| \\ &= |U| - |A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5| \\ &= |U| - \sum_{i \le 5} |A_i| + \sum_{i_1 < i_2 \le 5} |A_{i_1} \cap A_{i_2}| - \sum_{i_1 < i_2 < i_3 \le 5} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| \\ &\quad + \sum_{i_1 < i_2 < i_3 < i_4 \le 5} |A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap A_{i_4}| - |A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5|. \end{aligned}$$

Now we compute $|U| = 10!$, $|A_i| = 9!$ because one position is fixed, $|A_{i_1} \cap A_{i_2}| = 8!$ because two positions are fixed, etc. That is, the intersection of k subsets A_i has size $(10-k)!$. Also, there are $\binom{5}{k}$ such intersections. Thus,

$$\begin{aligned} &\big|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4 \cap \bar{A}_5\big| \\ &= 10! - \binom{10}{1} 9! + \binom{10}{2} 8! - \binom{10}{3} 7! + \binom{10}{4} 6! - \binom{10}{5} 5!. \end{aligned}$$

(9) [Points: 15] How many compositions of n [mistakenly printed as $[n]$ on the test] into four parts have no part equal to 3? (Assume $n \ge 12$.)

Solution. Another PIE problem. Let U be the set of all compositions of n into four parts. Thus, $|U| = \binom{n-1}{3}$. A composition looks like this: $n_1 + n_2 + n_3 + n_4 = n$, where all $n_i > 0$. We wish to avoid the four properties $n_1 = 3$, $n_2 = 3$, $n_3 = 3$, and $n_4 = 3$. Thus, let A_i be the set of compositions in which $n_i = 3$; then the desired number is

$$\begin{aligned}
&\left|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4\right| \\
&= |U| - |A_1 \cup A_2 \cup A_3 \cup A_4| \\
&= |U| - \sum_{i \le 4} |A_i| + \sum_{i<j\le 4} |A_i \cap A_j| - \sum_{i<j<k\le 4} |A_i \cap A_j \cap A_k| \\
&\quad + |A_1 \cap A_2 \cap A_3 \cap A_4|.
\end{aligned}$$

To compute $|A_1|$, note that if $n_1 = 3$ we are simply composing $n-3$ into three parts, $n_2 + n_3 + n_4 = n - 3$. The number of ways to do that is $\binom{n-4}{2}$. Thus, all $|A_i| = \binom{n-4}{2}$. There are 4 A_i's.

To compute $|A_1 \cap A_2|$ (or any $|A_i \cap A_j|$), if $n_1 = n_2 = 3$, we are simply composing $n-6$ into two parts; the number of ways to do so is $\binom{n-7}{1}$. For the same reason, any $|A_i \cap A_j| = \binom{n-7}{1}$. There are $\binom{4}{2} = 6$ such pairs.

To compute $|A_1 \cap A_2 \cap A_3|$, note that we are simply composing $n-9$ into one part: there is one way of doing this, since $n - 9 > 0$. Thus, any $|A_i \cap A_j \cap A_k| = \binom{n-10}{0}$. There are $\binom{4}{3} = 4$ such triples.

Finally, $|A_1 \cap A_2 \cap A_3 \cap A_4|$ is the number of ways to compose $n-12$ into no parts, none equal to three. Here we have to be more careful. As there are no parts, none of them can possibly equal 3, so we are counting compositions of $n - 12$ into no parts. Since $n - 12 \ge 0$, there are two cases. If $n = 12$, there is one way to compose $n - 12$ into no parts, i.e., take no parts, whose sum is 0. If $n > 12$, there are no ways to compose $n - 12$ into no parts, since $n - 12 > 0$.

The answer is therefore:

$$\begin{aligned}
&\left|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4\right| \\
&= \binom{n-1}{3} - 4\binom{n-4}{2} + 6\binom{n-7}{1} - 4\binom{n-10}{0} + \begin{cases} 0 & \text{if } n = 12, \\ 1 & \text{if } n > 12 \end{cases} \\
&= \begin{cases} \binom{11}{3} - 4\binom{8}{2} + 6\binom{5}{1} - 4 + 1 = 80 & \text{if } n = 12, \\ \binom{n-1}{3} - 4\binom{n-4}{2} + 6\binom{n-7}{1} - 4 & \text{if } n > 12. \end{cases}
\end{aligned}$$

(Either of these is a correct answer.)