Math 386 Quiz 1 October 14, 2022 Your name Solutions (corrected)

(1) (4 pts) State the Principle of Inclusion and Exclusion (PIE) for the case of a set $S$ (the "universe") and three properties, $P_1, P_2, P_3$, that each element of $S$ may or may not have. Let $A_i$ = the subset of $S$ that contains the elements with property $P_i$.

*Solution.* Either

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| =$$
$$|S| - \big(|A_1| + |A_2| + |A_3|\big) + \big(|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|\big) - |A_1 \cap A_2 \cap A_3|$$

or

$$|A_1 \cup A_2 \cup A_3| =$$
$$\big(|A_1| + |A_2| + |A_3|\big) - \big(|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|\big) + |A_1 \cap A_2 \cap A_3|.$$

(2) (8 pts) Let $M$ be the multiset $\{13 \cdot a, 13 \cdot b, 13 \cdot c\}$. Use the PIE to find the number of 30-combinations of $M$. State what your universe is and what your properties are. Numerical answers may be stated using binomial coefficients.

*Solution.* First state the universe and the properties. This is the first half of the answer.

Universe: $S =$ set of all 30-combinations of $M'$, where $M'$ is the multiset $\{\infty \cdot a, \infty \cdot b, \infty \cdot c\}$. (*Note*: The universe is not $M$ or $M'$. It consists of certain *sub*-multisets of the infinite multiset.)

Properties: $P_1$ is the property of an element of $S$ (that is, a 30-combination of $M'$) that it has $> 13$ $a$'s. $P_2$ is the similar property with $b$ instead of $a$. $P_3$ is the similar property with $c$ instead of $a$. (*Note*: These are the properties such that we want to avoid *all* of them.)

Now apply PIE (from Question 1). This is the second half of the answer. First, compute the set sizes.

$$|S| = (\text{number of 30-combinations of } M') = \binom{30+2}{2}.$$

$$|A_1| = (\text{number of 30-combinations of } M' \text{ with} \geq 14\ a\text{'s}) = \binom{16+2}{2},$$

because after you pick the required 14 $a$'s, you have to pick 16 more elements of the 30-combination.

$$|A_2| = |A_3| = |A_1|.$$

$$|A_1 \cap A_2| = (\text{number of 30-combinations of } M' \text{ with} \geq 14\ a\text{'s and} \geq 14\ b\text{'s}) = \binom{2+2}{2}.$$

$$|A_1 \cap A_2| = |A_1 \cap A_3| = |A_2 \cap A_3|.$$

$$|A_1 \cap A_2 \cap A_3| = (\text{number of 30-combinations of } M' \text{ with} \geq 14\ a\text{'s, } b\text{'s, and } c\text{'s}) = 0$$

because that adds up to more than 30 elements of the 30-combination. *Note*: Use the formula for combinations, not the formula for permutations.

Finally, apply PIE. Since we want *none* of the properties, we apply these numbers to the first form of the PIE. (If we wanted *at least one* of the properties, we would use the second form.)

$$\begin{aligned}
&|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| \\
&= |S| - \big(|A_1| + |A_2| + |A_3|\big) + \big(|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|\big) - |A_1 \cap A_2 \cap A_3| \\
&= \binom{30+2}{2} - 3 \cdot 16 + 22 + 3 \cdot \binom{2+2}{2} - 0 = 55.
\end{aligned}$$

You don't have to compute 55, but I like to do it to find out how big the answer is. I think 55 is surprisingly small, but that's only personal opinion.