Math 386 Quiz 3 November 16, 2022 SOLUTIONS

Why the name "generating function"? Because it is like a machine that generates one h_n after another, starting from h_0, then h_1, and continuing on as long as you keep cranking it.

(1) (4 pts) Simplify this infinite series: $f(x) = \sum_{n=0}^{\infty} \binom{n}{2} x^n$.

Using the substitution $m = n - 2$ and Newton's Binomial Theorem,
$f(x) = 0 + 0 + \sum_{n=2}^{\infty} \binom{n}{2} x^n = \sum_{m=0}^{\infty} \binom{m+2}{2} x^{m+2} = x^2 \sum_{m=0}^{\infty} \binom{m+2}{2} x^m = x^2(1-x)^{-3}$.

In order to solve this problem you have to know Newton's Binomial Theorem backwards, so you can go from an infinite series to the fraction. You have to recognize that the denominator exponent is 3, from seeing the 2 in the binomial coefficient $\binom{n}{2}$.

Note that you *never* substitute a value for x, even if it gives a number. If you wrote $f(1), f(2)$, etc., or something similar, you were off in the wrong direction.

(2) (4+4 pts) We want to make a fruit basket of n fruits. The basket should have an even number of Apples (0 is an even number), an odd number of Bananas, and at most 2 Cherries. Let h_n = the number of ways to make such a basket.

(a) Construct the generating function for h_n.

In a problem like this we assume all Apples are essentially the same, and likewise for Bananas and Cherries.

The generating function for Apples is $1 + x^2 + x^4 + \cdots + x^{2k} + \cdots$. There is one way to pick 0 apples (hence $1x^0$), no way to pick 1 apple (hence $0x^1$), 1 way to pick 2 apples (hence $1x^2$), etc. That is, the Apple g.f. is $1x^0 + 0x^1 + 1x^2 + 0x^3 + 1x^4 + \cdots$.

The generating function for Bananas is $x + x^3 + x^5 + \cdots + x^{2k+1} + \cdots$. There is no way to pick 0 bananas (hence $0x^0$), 1 way to pick 1 banana (hence $1x^1$), no way to pick 2 bananas (hence $0x^2$), etc. That is, the Banana g.f. is $1x^0 + 0x^1 + 1x^2 + 0x^3 + 1x^4 + \cdots$.

The generating function for Cherries is $1 + x + x^2$. There is one way to pick 0 cherries (hence $1x^0$), one way to pick 1 cherry (hence $1x^1$), 1 way to pick 2 cherries (hence $1x^2$), and no ways to pick more cherries (hence, $0x^3 + 0x^4 + \cdots$). That is, the Cherry g.f. is $1x^0 + 1x^1 + 1x^2$.

The g.f. for the combined basket is obtained by multiplying the three fruit g.f.'s, that is, it is

$$(1 + x^2 + x^4 + \cdots + x^{2k} + \cdots)(x + x^3 + x^5 + \cdots + x^{2k+1} + \cdots)(1 + x^1 + x^2)$$

or more simply

$$(1 + x^2 + x^4 + \cdots)(x + x^3 + x^5 + \cdots)(1 + x^1 + x^2).$$

Note *"generating function"*. Do not try to find h_n. That is a different problem.

(b) Simplify the generating function as much as possible.

Using the geometric series formula we get

$$\frac{1}{1-x^2}\frac{x}{1-x^2}(1 + x^1 + x^2) \text{ or } \frac{x}{(1-x^2)^2}\frac{1-x^3}{1-x} \text{ or a similar expression.}$$