(1) (10 points) Give a combinatorial proof of Pascal's identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, where $n > k > 0$.

Solution. Let S be a set of n elements and let a be one element of S. The left side, $\binom{n}{k}$, counts all k-element subsets T of S. Some of those subsets T contain a and some do not. The ones that do not contain a are precisely the subsets of $S - a$, which has $n - 1$ elements, so there are $\binom{n-1}{k}$ of that type. The subsets T that do contain a consist of a and also $k-1$ elements of $S-a$, so the number of them equals the number of $k-1$-element subsets of $S - a$, which is $\binom{n-1}{k-1}$. Since there are $\binom{n-1}{k}$ k-element subsets that do not contain a and there are $\binom{n-1}{k-1}$ that do contain a, the total number of k-element subsets of S equals $\binom{n-1}{k} + \binom{n-1}{k-1}$, the right side. Since the left and right sides both count all k-element subsets of S, they are equal.

(2) (5 points) Express the sum $\binom{n}{n} + \binom{n+1}{n} + \cdots + \binom{n+5}{n}$ as a single binomial coefficient. Use any method. Proof is not required.

Solution. $\binom{n+6}{n+1}$. Here is a proof:

$$\begin{aligned}
&\binom{n}{n} + \binom{n+1}{n} + \binom{n+2}{n} + \binom{n+3}{n} + \binom{n+4}{n} + \binom{n+5}{n} \\
&= \binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \binom{n+3}{3} + \binom{n+4}{4} + \binom{n+5}{5} \\
&= \binom{n+6}{5}
\end{aligned}$$

by one of the important formulas for binomial coefficients, (5.18) with $r = n$ and $k = 5$. A simpler way (opinion!) is to use Pascal's identity several times after two small adjustments, as here:

$$\begin{aligned}
&\binom{n}{n} + \binom{n+1}{n} + \binom{n+2}{n} + \binom{n+3}{n} + \binom{n+4}{n} + \binom{n+5}{n} \\
&= \binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \binom{n+3}{3} + \binom{n+4}{4} + \binom{n+5}{5} \\
&= \binom{n+1}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \binom{n+3}{3} + \binom{n+4}{4} + \binom{n+5}{5} \\
&= \binom{n+2}{1} + \binom{n+2}{2} + \binom{n+3}{3} + \binom{n+4}{4} + \binom{n+5}{5} \\
&= \binom{n+3}{2} + \binom{n+3}{3} + \binom{n+4}{4} + \binom{n+5}{5} \\
&= \binom{n+4}{3} + \binom{n+4}{4} + \binom{n+5}{5} \\
&= \binom{n+5}{4} + \binom{n+5}{5} \\
&= \binom{n+6}{5}.
\end{aligned}$$

(3) (5+5+bonus points) There are 99 bottles of beer on the wall,[1] all in a row.

(a) How many ways are there to pick 12 of the bottles?
Solution. $\binom{99}{12}$. (No explanation is needed for this.)

(b) How many ways are there to pick 12 bottles, if you are not allowed to pick any two adjacent bottles? (That is, any two you pick must be separated by at least one bottle.)
Solution. There are several ways to solve this. I will use the most flexible one.

Suppose our 12 chosen bottles have x_0 bottles (not chosen) on their left, there are x_i bottles between the ith bottle and the next one (for $i = 1, 2, \ldots, 11$, and there are x_{12} bottles to the right of the last chosen bottle. Hence, $x_0 + x_1 + \cdots + x_{11} + x_{12} = 99 - 12 = 87$.

The conditions of this problem require $x_0, x_{12} \geq 0$ and $x_1, \ldots, x_{11} \geq 1$. So, we replace some of the variables: let $y_i = x_i - 1$ for $i = 1, 2, \ldots, 11$. Then the restriction is that $y_i \geq 0$. Now we have $x_0 + (y_1 + 1) + \cdots + (y_{11} + 1) + x_{12} = 87$, which simplifies to $x_0 + y_1 + \cdots + y_{11} + x_{12} = 76$, with all variables ≥ 0. The number of solutions is $\binom{76+13-1}{13-1} = \binom{88}{12}$. That is the number of ways to pick the 12 bottles.

(c) How many ways are there to pick 12 bottles, if any two you pick must be separated by at least k bottles, where $k \geq 1$ is any integer? Carefully consider different possible answers depending on the value of k, which may be any positive integer, small, medium, or large.
Solution. The solution method is the same as in part (b); only the numbers are different. Here each $x_i \geq k$ for $i = 1, 2, \ldots, 11$ so I replace x_i by $y_i = x_i - k$ for $i = 1, 2, \ldots, 11$. The equation becomes $x_0 + (y_1 + k) + \cdots + (y_{11} + k) + x_{12} = 87$, simplifying to $x_0 + y_1 + \cdots + y_{11} + x_{12} = 87 - 11k$, all variables ≥ 0.

Now it gets a bit complicated. If the constant term $87 - 11k \geq 0$, our general method gives the answer: there are $\binom{(87-11k)+13-1}{13-1} = \binom{99-11k}{12}$ ways to pick the bottles. However, if the constant term is negative, then it's impossible to satisfy the requirements; since all variables are ≥ 0, their sum can't be negative. Therefore, the answer is 0. The number of ways to pick the bottles is therefore

$$\begin{cases} \binom{99-11k}{12} & \text{if } k \leq 87/11, \text{ i.e., } k \leq 7, \\ 0 & \text{if } k \geq 8. \end{cases}$$

Notice that the first formula gives the wrong answer if $k \geq 8$, because the upper number in the binomial coefficient is negative; the value of the coefficient will not be 0. Remember our general definition of a binomial coefficient!

I gave 5 bonus points for solving the question without noticing that there are two different cases, 8 points for a complete solution.

(4) (10 points) Let f_n be the number of ways to perfectly cover a $2 \times n$ board by dominoes. Prove that $f_{n+1} = f_n + f_{n-1}$ for $n \geq 2$.
Solution. It helps to draw a picture of the board. Make it 2 squares high and $n + 1$ wide. Name the two left-most squares: the upper left square is Xenia (X) and the lower left square is Yannakis (Y).

[1]Taken from a classic day camp song.

We have to cover X with a domino. The domino can be horizontal or vertical. If the domino on X is vertical, it also covers Y and the remaining board is $2 \times n$, which can be covered in f_n ways. If the domino on X is horizontal, it extends to the right and the only way to cover Y is by another horizontal domino, also extending to the right. These two dominos cover two columns, leaving an empty $2 \times (n-1)$ board, which can be covered in f_{n-1} ways. Every cover of the original board is counted in exactly one of these two cases, so f_{n+1}, the total number of ways to cover the original board, equals $f_n + f_{n-1}$.

(5) (3+3+5 points) Consider the multiset $M = \{14 \cdot a, 15 \cdot b, 16 \cdot c\}$.
Solution. You have to know the difference between combinations and permutations.

(a) How many 45-combinations are there of M?
Solution. 1. Because $|M| = 45$, it is its own only 45-combination. Partial credit for a complicated answer derived by PIE, which should $= 1$ but I didn't try to simplify it.

(b) How many 44-combinations are there of M?
Solution. 3. Because $44 = |M| - 1$, we take one element out of M to get a 44-combination. Since there are only 3 distinguishable elements of M, there are 3 44-combinations.
Solution. (Second way.) We remove a 1-combination from M. The number of 1-combinations is 3.

(c) How many permutations are there of M?
Solution. $45!/14!15!16!$.

(6) (14 points) How many 28-combinations are there of the multiset M in problem (5)?
Solution. This problem requires the Principle of Inclusion and Exclusion.

Part of the solution is the set-up, including choosing a universe. In this kind of problem, first we define $M^* = \{\infty \cdot a, \infty \cdot b, \infty \cdot c\}$. Then we can define the universe, $U = \{\text{all 28-combinations of } M^*\}$. Then we define the properties of a 28-combination C of M^*. (In this problem we are avoiding all properties, which are that a 28-combination has too many of some element of M.) So:

$P_1 =$ the property that C has more than 14 a's.
$P_2 =$ the property that C has more than 15 b's.
$P_3 =$ the property that C has more than 16 c's.

This gives the sets for exclusion:

$A_1 =$ the set of 28-combinations of M^* that have ≥ 15 a's.
$A_2 =$ the set of 28-combinations of M^* that have ≥ 16 b's.
$A_3 =$ the set of 28-combinations of M^* that have ≥ 17 c's.

(If you omit the properties but give the sets, it's okay.)

Now we do the calculations. We know $|U| = \binom{28+3-1}{3-1}$. As for A_1, a 28-combination in A_1 has 15 a's and $28 - 15 = 13$ additional arbitrary elements of M^*, so $|A_1| = \binom{13+3-1}{3-1}$. Similarly, $|A_2| = \binom{12+3-1}{3-1}$ and $|A_3| = \binom{11+3-1}{3-1}$.

Next, an element of $A_1 \cap A_2$ has to be a 28-combination with ≥ 15 a's and ≥ 16 b's, which is impossible; therefore $A_1 \cap A_2$ is empty, i.e., $|A_1 \cap A_2| = 0$. For similar reasons, $|A_1 \cap A_3| = 0$ and $|A_2 \cap A_3| = 0$, and therefore $|A_1 \cap A_2 \cap A_3| = 0$.

Finally, we put these numbers into the PIE formula. We are avoiding the properties so the formula to use is

$$
\begin{aligned}
\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 &= |U| - (|A_1| + |A_2| + |A_3|) \\
&\quad + (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) - |A_1 \cap A_2 \cap A_3| \\
&= \binom{28+3-1}{3-1} - \left[\binom{13+3-1}{3-1} + \binom{12+3-1}{3-1} + \binom{11+3-1}{3-1}\right] \\
&\quad + (0+0+0) - 0 \\
&= \binom{30}{2} - \left[\binom{15}{2} + \binom{14}{2} + \binom{13}{2}\right].
\end{aligned}
$$

Note of explanation: The universe can't be M^* (and it can't be the set of 28-combinations of M). The universe has to be a set of 28-combinations (of something) because in PIE we count some of the elements of the universe while excluding others. The universe also has to be easy to count. The 28-combinations of M^* are easy to count because there is no limit on how many of each element we can have in a 28-combination. The 28-combinations of M are hard to count.

(7) (10 points) How many solutions in integers does the following equation have, with the stated restrictions on the variables?

$$
x_1 + x_2 + x_3 + x_4 + x_5 = 77,
$$
$$
x_1 \geq 0,\ x_2 \geq 15,\ x_3 \geq 15,\ x_4 \geq 15,\ x_5 \geq -10.
$$

Solution. Normalize the variables to be ≥ 0. So, I will let $y_1 = x_1$, $y_2 = x_2 - 15 \geq 0$, $y_3 = x_3 - 15 \geq 0$, $y_4 = x_4 - 15 \geq 0$, $y_5 = x_5 - (-10) \geq 0$. Carefully substituting for the x's in the equation, I find that

$$
y_1 + (y_2 + 15) + (y_3 + 15) + (y_4 + 15) + (y_5 - 10) = 77,
$$

which simplifies to $y_1 + y_2 + y_3 + y_4 + y_5 = 42$ with all $y_i \geq 0$. The number of solutions is $\binom{42+5-1}{5-1}$.

Notice that there is no upper bound on any of the variables. That is why we don't need the PIE.

(8) (10 points) Prove that, if you have 6 people, and every pair of them either loves or hates each other, then there is a triple who all love each other or all hate each other. (Do not test this at home.)

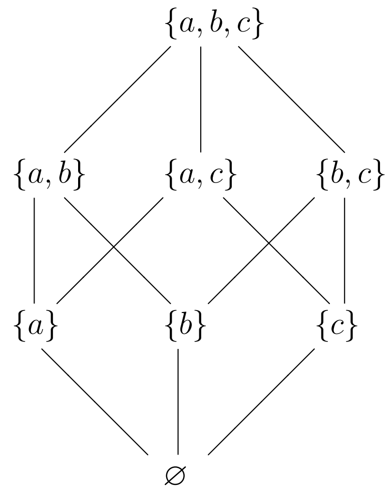
Solution. Look at one of the 6 people, say Alfred. Alfred has 5 relationships of two "colors", love and hate. By the Pigeonhole Principle, Alfred loves 3 (at least) or hates 3 (at least). Suppose Alfred loves 3. If any two of those three love each other, then with Alfred they make a loving triple. If no two of those three love each other, they make a hating triple by themselves. Thus, in this case the claimed triple exists. If on the other hand Alfred hates 3, the same reasoning shows there is a hating triple including Alfred or a loving triple without him. In every case, therefore, the claimed triple exists.

Trying to solve the problem by somehow classifying pairs doesn't work. I think that shows how useful the PHP is.

(9) (4+3+3 points) For the partially ordered set $\mathcal{P}$ of all subsets of $\{a, b, c\}$:

(a) Draw the Hasse diagram.

Solution.



(b) Find a maximal chain in $\mathcal{P}$.

Solution. Many, such as $\varnothing \subset \{a\} \subset \{a, c\} \subset \{a, b, c\}$, or you can write it as $\{\varnothing, \{a\}, \{a, c\}, \{a, b, c\}\}$ (or other ways).

(c) Find an antichain of at least 3 elements of $\mathcal{P}$.

Solution. Two exist: $\{\{a\}, \{b\}, \{c\}\}$ and $\{\{a, b\}, \{a, c\}, \{b, c\}\}$.

(10) (10 points) Find the number of permutations $i_1 i_2 i_3 i_4 i_5$ of the numbers $1, 2, 3, 4, 5$ such that

$$i_1 \neq 1, 2, \;\; i_2 \neq 1, 2, \;\; i_3 \neq 4, \;\; i_4 \neq 4, 5.$$

Solution. Draw the 5×5 board with the forbidden squares x'd out. Then calculate r_1, r_2, r_3, r_4, r_5 by using the x'd squares. The values are $r_1 = 7$, $r_2 = 15$, $r_3 = 10$, $r_4 = 2$, $r_5 = 0$. The number of permutations is therefore $5! - 7 \cdot 4! + 15 \cdot 3! - 10 \cdot 2! + 2 \cdot 1! - 0 \cdot 0! = 24$.

To compute r_2, for instance, a good method is to notice that the forbidden board has two separate parts: a 2×2 square in the upper left and three squares in an L shape in the lower right. To get r_2 we count the number of placements of 2 nonattacking rooks in the upper left portion (this is 2), the number in the lower right portion (this is 1), and the number of ways to put one rook in each portion (this is 4×3), giving the total of $2 + 1 + 12 = 15$.